Linear Algebra

Tuesday, September 19, 2006
7:50 AM

Course webpage: Blackboard www.ea1.mccormick.northwestern.edu
- course info 3 links gen. info.

4 lectures / week Tues–Fri.
Some Mon off, some lab (comp. lab) MG47

Quizes Oct 12 Oct 26 Nov 16
At 7 PM

Final exam on Tues Dec 5 1500-1700

Weekly HW due through course management site
Midnight Wed deadline

There's a course outline & reading info section – WHERE HW
Case studies - Economics, encryption, digital audio
Course Managment Systems

Linear Algebra - Study of vectors and their transformations.

Vector - element of a vector space.

3 equations, 3 unknowns, solve. And how that applies to other stuff.

EA 2 & 3 - More linear Algebra, statics, dynamics, physics

Sectors of economy

A: Manufacturing

B: Energy

C: Transportation

Final demand (external)

A: 5 units

B: 10 units

C: 8 units

How much should each sector produce?

Intermediate demand - for a sector to produce their final demand, they must produce units to satisfy each other.

More data is necessary.

To produce 1 unit of A output, we use input $\begin{cases} .1 \text{ A units} \\ .2 \text{ B units} \end{cases}$

To produce 1 unit of A output, we use input $\begin{cases} .1 \text{ A units} \\ .2 \text{ B units} \\ .3 \text{ C units} \end{cases}$

Do research on other 2 sectors

$$\begin{array}{c|ccc} & \underline{A} & \underline{B} & \underline{C} \\ \hline A & 0.1 & 0.1 & 0.3 \\ B & 0.2 & 0.1 & 0.06 \\ C & 0.3 & 0.2 & 0.1 \end{array} \leftarrow \text{output} \qquad \begin{matrix} X_A \\ X_B \\ X_C \end{matrix}$$

input

consumption matrix

Vectors, Column Vectors

Wednesday, September 20, 2006
6:56 AM

output

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0.1 & 0.1 & 0.3 \\ B & 0.2 & 0.1 & 0.06 \\ C & 0.3 & 0.2 & 0.\ \end{array}$$

consumption matrix C

input

intermediate demand

A	5 units	Man.
B	10 units	Energy
C	8 units	Transp.

Unknowns

X_A = # of units produced by A

X_B " by B

X_C " by C

We want each sector to produce the total demand to maximize the profit.

Production Level = intermediate demand + final demand

$$X_A = 0.1X_A + 0.1X_B + 0.3X_C + 5$$
$$X_B = 0.2X_A + 0.1X_B + 0.06X_C + 10$$
$$X_C = 0.3X_A + 0.2X_B + 0.1X_C + 8$$

Linear equations b/c they form straight lines, and this is b/c there are no exponents greater than 1

<u>Vector</u> – Ordered list of numbers

The order matters – 3,5,9 is not the same as 5,9,3

Column Vector

$$\begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix} \quad \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

belong to $\mathbb{R}^3$

$\underbrace{\mathbb{R}^3}_{\text{Real \#s}}$ } ordered list of 3 real #s

$\mathbb{R}^2$

$\mathbb{R}^4$

We can multiply vectors by scalars

Scalar Multiplication

$$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Scalar

Vector Addition

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}3\\2\\0\end{bmatrix} = ??$$ NOT DEFINED

Linear Combination

$$3\begin{bmatrix}1\\2\\1\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}5\\6\\7\end{bmatrix}$$

coefficients, "weights"

put weights into a vector

$$\begin{bmatrix}3\\4\\2\end{bmatrix}$$

Same computation:

$$\underbrace{\begin{bmatrix}1&0&1\\2&0&0\\1&1&0\end{bmatrix}}_{\substack{\text{Matrix}\\A}}\underbrace{\begin{bmatrix}3\\4\\2\end{bmatrix}}_{\substack{\text{Vector}\\b}} = \underbrace{\begin{bmatrix}5\\6\\7\end{bmatrix}}_{\substack{\text{Multiplication}\\Ab}}$$

# of columns of A must be the same as the # of entries in b.

The matrix is always first, on the left

$$\begin{bmatrix}1&0&2&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

$$\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}3\\5\\2\end{bmatrix} = ??$$ MAKES NO SENSE

Final DEMAND

$$\begin{bmatrix}5\\10\\8\end{bmatrix}$$

$\mathbb{R}^3$

Production Levels

$$X = \begin{bmatrix}x_A\\x_B\\x_C\end{bmatrix}$$

Consumption Matrix

rows, colum

3×3

$$C = \begin{bmatrix}0.1&0.1&0.3\\0.2&0.1&0.06\\0.3&0.2&0.1\end{bmatrix}$$

$$x = \underbrace{Cx}_{\text{intermediate demand}} + d \qquad x - Cx = d$$

Matrix Addition

$$\begin{bmatrix}1&0\\1&0\end{bmatrix} + \begin{bmatrix}3&2\\1&0\end{bmatrix} = \begin{bmatrix}4&2\\2&0\end{bmatrix}$$

$$(A_1 + A_2)b = A_1 b + A_2 b$$

We need to write x as $A_1 x$. C would be A_2.

Identity Matrix, Variables

Thursday, September 21, 2006
6:58 AM

$x = Cx + d$ ☑ HW on Blackboard due Wed

Cx = linear combination $x, d \in \mathbb{R}^3$ $x = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$ unknowns

Matrix(vector)

Result = intermediate demand Vectors in $\mathbb{R}^3$

$d = \begin{bmatrix} 5 \\ 10 \\ 8 \end{bmatrix}$

C is 3×3 Matrix

$x - Cx = d$

Matrix

$Ix - Cx = d$ vector

$(I - C)x = d$

$\boxed{A \quad x \quad = b}$

Standard form

$x = x_A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_C \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$x = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{identity matrix "I"}} x$

$$\boxed{\begin{array}{l} A \text{ is } m \times n \\ x \in \mathbb{R}^n \\ b \in \mathbb{R}^m \end{array}}$$

b is a linear combination of A, so it has m entries since A has m rows.

Matlab !!! Start playing w/ it.

Variable $\boxed{\overset{x}{\text{DATA}}}$ $\boxed{\overset{x}{\begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}}}$ In Matlab, we create variables

Names start w/ letters
Case sensitive
this_variable_works. But that's a long name for a variable.

assignment statements

:) $x = 3$ $3 = x$ ERROR!

variable on left

data on right

$x = 3 + x$ $\boxed{\overset{x}{3}}$ this assignment says "take the

$x = 3 + x$ $\quad$ $\overset{x}{\boxed{3}}$ this assignment says "take the previous variable and add 3" so if the variable was 3, the new variable would be 6.

Also $x = x + 3$

Command Window is where you can enter in commands 1 by 1 and Matlab will execute them.

CLEAR <u>variable name</u> it gets rid of the variable.

CLEAR <u>ALL</u> gets rid of everything

* means multiplication

^ means exponent

; means Matlab won't tell you that it got the command but it will carry out what it was told.

you don't need spaces, it just makes it easier to read

x = [3 ; 5 ; 2] tells Matlab that x is a vector

x = [.1 .1 .2 ; .3 .0 .1 ; .3 .11 .03] 3x3 Matrix

i = [1 0 0 ; 0 1 0 ; 0 0 1] identity matrix.

<u>OR</u>

i = eye(4) gives us a 4x4 identity matrix

Colon Operators, Subarrays

Friday, September 22, 2006
7:02 AM

Quiz Options — 5 - 6:30 pm
Oct. 26th — 6 - 7:30 pm if Chem moves to 8 PM
Thurs — 8:30 - 10 pm X-hockey, doesn't work
Oct. 12th
is another quiz
Nov. 16th
is yet another

Input
Data representation
1x1 is a scalar, and is the smallest unit in Matlab
same as 4 or ['a']

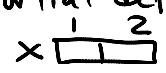
You give names to scalars & matrices, and those are variables

Functions - later. Solve Prob.
Display / Save results
- integer values displayed as integers.
- doubles displayed as w/ 4 decimal digits
  2.45678
  2456.78
- doubles that are too large or too small are displayed in scientific notation.

Assignments.
radius = 15.5 you don't have to declare variables
radius_copy = radius;
circumfrence = 2*pi* radius pi is predefined.

Predefined variables
pi ; ans (stores result) ; i, j ($\sqrt{-1}$) ; Inf (infinity)
NaN (doesn't work) ; eps; date; clock.

More assignments
x(1,2) = 5; % partial def. - all other values filled w/ 0.



y(4) = 0.1 since y is already assigned, say, as $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ it will replace

row-major - we store the #1 in the 1st row, then #2, #3, #4, & #5 & 6

column major - means #1 in the column, then #2, 3, 4, 5, 6

The matrix we specified is column major. It will replace the 2 with 0.1.

$$data = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

data(2) = 9, changes the 4 into a 9.

d2 = [data data] takes 2 copies of the matrix, puts them side-by-side, and makes one big matrix

d3 = data' % transpose $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T$ transpose does this: $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

dir1 = [2 3]' shorthand

dir2 = [2 3] % same as dir1

I = eye(4) 4x4 identity matrix

I = eye(2, 4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

>> ones(4,5)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

format compact means save a few lines.

size(ans) gives you the size of the matrix

if you did x = ones(4, 5);

y = zeros(size(x)) So you can be lazy.

Colon Operator

Shorthand notation that allows you to generate

d = 1:3

you get d =

1 2 3

d = a:c

you get d =

#s between a & c

d = 'a':'c'

you get d =

abc

size(d) =

1 3 b/c Matlab thinks abc is a 1x3 matrix

x=(1:0.1:2)*2 means you take everything between 1, 0.1, & 2 and make a matrix, then multiply by 2

y=[x;x*2] Matrix

x = 1:0.1:2; y = [x;x*2];

1 1.1 1.2 1.3
2 2.2 2.3 2.4 ...

Subarrays You can use the colon operator to access chunks of what you're working on.

x=(1:3) is (10, 20, 30) if x originally was [10, 20, 30, 40]

x=(1:2:4) is (10, 30)

x=(1:) means get everything in the 1st row

Entering Data

Tuesday, September 26, 2006
7:01 AM

- Way to enter data: shortcuts, :
  - Way to refer to data so we can use it in calculations
  - Way to perform the calculations
  - Way to display results

$x = Cx + d$
$x - Cx = d$
$(I - C)x = d$

<u>eye(size(C))</u>

$(I-C)^{-1}(I-C)x = (I-C)^{-1}d$
$Ix = (I-C)^{-1}d$
$x = (I-C)^{-1}d$

x = input('please enter number x:');
This prompts the user to type something in, and then assigns what they type to x.
x = input('Please enter table x:');
name = input('Please enter your name':);

MAT-file preserves variables names, types, sizes
can only be read by
MATLAB typical ext = .mat

ASCII file data saved as ASCII strings separated by spaces.
variable info lost
can be read by any program/human

Loading files: load filename.mat
load filename.dat ASCII
load filename.mat x y only those variables

inv(x) is the inverse of x.

array operations + matrix operations
↓ Do not always produce

element by element basis
& between matrix & scalar

... not always produce
the same result.

$$? \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} ?0 & ?1 \\ ?2 & ?3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}$$ Array operation

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix}$$ Standard multiplication.

$(1\cdot 1) + (2\cdot 3) = 7$ 1st row 1st column
$(1\cdot 2) + (2\cdot 4) = 10$ 1st row 2nd column
$(0\cdot 1) + (1\cdot 3) = 3$ 2nd row 1st column
$(0\cdot 2) + (1\cdot 4) = 4$ 2nd row 2nd column

$(n \times m)(m \times p) = n \times p$?

.* is array multiplication
* is multiplication.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 1 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 7 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} .* 2 = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 7 & 5 \end{bmatrix} .* \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 28 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} ./ 2 = \begin{bmatrix} .5 & 1 \\ 1.5 & 2 \end{bmatrix}$$

$$2 ./ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ .67 & .5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 7 & 5 \end{bmatrix} ./ \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & .3 \\ 1.75 & 1 \end{bmatrix}$$

$A / B = A(B^{-1})$ standard

$A .\backslash B = B / A$ array

or

$A \backslash B = B / A$ matrix

So we have soln. x=[inv(eye(size(C)) - C)]*d
disp(x) shows x.

msg = ['the production vector is']

disp (mgoy) ; disp (x)

the production vector is

number 1 x_m

number 2 x_A

number 3 x_s

# of spaces

fprintf (Agr production is %4.2f) , x(1)

# of x's displayed like

fprintf (' % 4.2 f , %4.2f,' x(1), x(2)) 1.22, instead of 1.2257987

Plotting in MATLAB

Wednesday, September 27, 2006
6:59 AM

num2str is how you create a string.
floor(x) is the biggest integer smaller than x.
ceil(x) is the smallest integer larger than x.
cos(x), sin(x), sqrt(x)

max (4 6 6) gives you the largest # in a matrix.
= 6

plotting w/ Matlab – get data, (like record highs + lows for Sept.)
legend needed for once.
2 plots on the same figure
Diff color for each one.

- Create 2 vectors - one for x values, one for y values.
  - Must have same length.
- "plot" is the function
  plot (x,y) x.^2 b/c 2 is a scalar.
  plot y as a function of x x is a vector
  w/ Linespec being a string
  that describes the style (e.g. dotted), color, & marker
  symbol of the lines. "plot(x, y, linespec)"
  eg. x = [-5 : 1 : 5]; y = x.^3; plot(x,y,'bo--');
  b for blue o for circular markers, -- for dashed line.
- "plot (x1, y1, linespec1, x2, y2, Linespec 2)"
  plot both functions in the same figure.
- "plot (... 'Property Name', Property Value, ...)
  properties ex. line width, etc.
  plot (x, y1, 'bo--', x, y2, 'k*:', 'Linewidth', 2);
  You can't have 2 'Linewidths' in the same command. This is not the method for putting properties in the same figure.
- "loglog" is similar to "plot", but used for exponential functions.
- "xlabel('some text')"
- "ylabel('some text')"
- "legend('explanation for x', 'exp for y')

- "title ('some text')
- toggle grid lines by 'grid on' or 'grid off'

M-files are files you upload, usually .txt files, and Matlab executes the files for you. Could be typical script or function.

.mat file is a Matlab file, can only operate using Matlab.

load "name.mat"

"hold;" means for the file to stay, for the graph to not be deleted and for Matlab to plot your following line on the same graph.

'Legend ("High", "low")

'Legend ('High', 'Low', 0)

↑ position.

"grid" gives you a grid.

" xlabel ('days'); ylabel ('temp');

To Matlab help 'Axes' or 'Axis' tells you about this.

" title ('Highs and Lows. Source: weather.com')

↑ we want to create a 2nd line here. So insert ' on end of 'Highs & Lows' and 9, and 10 and insert ' on beginning of 'Source...' 10 is 'next line'. And insert brackets on inside of ( ).

to save this variable information

"save data.mat specific_variable1 specific_variable2"

or just this

'%d' is for integers

Iterative Method (And the inception of If/Then, Book 2)

Thursday, September 28, 2006
6:58 AM

x_0 - start off w/ this

$x = [\underbrace{Cx}_{\text{intermediate demand}} + d]$ external demand

NO equilibrium if x on right $= x_0$. $x^{(0)}$ gives you $x^{(1)}$

Now if you want to approximate again, you get

$$x^{(2)} = C x^{(1)} + d$$
$$x^{(3)} = C x^{(2)} + d$$
$$x^{(4)} = C x^{(3)} + d$$
$$\boxed{x^{(n)} = C x^{(n-1)} + d}$$

you end up w/ an approximation that's very close to the real thing.

$x^* = Cx^* + d$

Iterative Method

Execute 1 + calculations repeatedly until some condition has been satisfied.

compare x's - if the diff between them is really small they are just about the same.

↓

print results

do it again ← if not small enough

x = input( )
y = input( )
if y is 0, then fprintf ("division by zero not allowed")
otherwise, (x / y).

x ==

check whether x has the same value as y

1 == 5
0 = false in Matlab.
'a' == 'a'
1 = true
[10 1 2] == [10 3 -2]
[1 0 0] because 1st elmt. is true & other 2 are false
'a' == 'A'

If/Then - Sequel to Either/Or?
Knowledge = Power

$x \sim= y$ test inequality
0 = equal
1 = not equal

Boulean values
True/False Values.
Allow us to create simple expressions

$x < y$ less than
$x <= y$ less than/equal to
$x > y$ greater than
$x >= y$ greater than/equal to

relational operators.

if today == 'Sunday'
then weather == 'sunny'
have a picnic
else sleep in.

We want to combine them
They both have boulean values.

logical operators — && : and

supports short circuit evaluation
ONLY WORKS FOR SCALARS

short circuit evaluation

$(x <= 5) \&\& (y == 10)$
T && T → T, 1
everything else → 0

Logical operators — && : and
& : and

x || y
x | y
if both x, y are 0 → result 0
all other cases → result 1

if t = sunday && w = sunny
then picnic
else sleep

if t = sunday | w = sunny
then picnic
else sleep

diff is that if it's Sun, then we have a picnic no matter what the weather is. But if it's false, we check the weather and we get a picnic if it = sunny.

Xor inclusive or
xor (x, y)
this is the Either/Or operator.
"Xor"

x	y	result
1	0	1
0	1	1
0	0	0
1	1	0

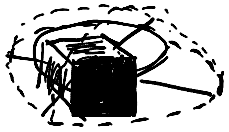
~ NOT operator → uniary operator

x, ~x the opp. of x

if~(tank = = 'full') if the ( ) expression is true, the ~ negates it. same as if (tank ~= 'full')
fill up
fill up.

Relational: = =, ~=, <, <=, >, >=

Logical && , & , ||, | , xor, ~

If/Then book 

x || y && z+z < 4 ||

x + $\boxed{y * z}$ PEMDAS

x + y − z Left-right

ischar(A) is character

isfloat

isinteger

isinf is infinity

isnumric(A) array/matrix

logical (x) convert # into boolean.

4 || 0

True — 1, b/c if it's not 0, it's true.

If, Else, End

Friday, September 29, 2006
7:01 AM

$$f(x) = \begin{cases} x^2, & \text{if } 1 <= x <= 10 \\ x-1, & \text{other value} \end{cases}$$

```
x = input('Please enter x')

if 1<=x<=10
 true/   \False
y=x^2     y=x-1
    \    /
     disp(x)
```

```
if x >= 1 & x <= 10
    y = x.^2
else
    y = x-1
end
disp(y)
```

```
IF condition
    stat1
    stat2
    stat3
end
```

```
if tank='empty'
true/        /false
fill up
    \ end
     drive
```

```
if score > 90
    grade = 'A'
elseif score > 80
    grade = 'B'
else if score > 70
    grade = 'C'
else
    grade = 'F'
end
```

equivalent of having more if statements in the else statement.

NUMB3R5

```
SWITCH
 if x==y1
 then stmnt 1
 else if x==y2
 then
   stmnt2
 else if x==y3
 then
   stmnt3
else
```

→

```
switch(x)
case y1
    stmnt1
case y2
    stmnt2
case y3
    stmnt3
otherwise
    stmnt4
```

x must be a character for this to work. or a #.

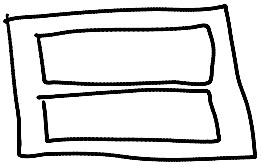
```
else
   stmnt4
end
```

```
stmnt4
end
```

Back to plotting

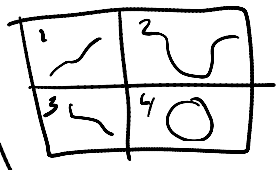
figure(1) go on w/ plot commands > same window
figure(2) puts another plot in.

subplots



2x1 table.

subplot(2,1)
subplot(2,2,4)
plot(...)



2 x 2 table

tells you to start w/ 4th quadrant. Not like math.

axis([xmin xmax ymin ymax])

0 31
52 100

specify differently.

default: 0 100

xlabel('Day') xlabel('\bf\'fontsize {12} Day')

boldface

The program sometimes calculates things incorrectly even though they look right. The easiest way to fix it is to do it on paper, but the best way to do it is to get Matlab to figure it out for you. If you tell Matlab to do your code line-by-line, you can find the statement that caused the stray value.

Debugger 'step'
  step in (a function)
  step out
Breakpoints - execute everything up to a point and then wait

While, For, Check

Tuesday, October 03, 2006
7:00 AM

M-file - you can create a file and then insert it into Matlab. It'll work only on matlab. You have to click on New → M-file. Then save

Expression is anything that has a value. sin(x) is an expression. If you're asked to write an expression about 2 matrices, you can make them a & b and leave it at that - you don't need a specific a or b.
A == B is usually found in if/then statements.

$X = C * X + d \qquad x^0 = d$
$x^1 = C * d + d$
$x^2 = C * x^1 + d$
$x^3 = C * x^2 + d$
until $x^n = x^{n-1}$

initialize x
check if diff is small
↓
condition
↓
if true, print
if diff is not small enough
↓
perform calculation
↓
compute diff
↓
repeat. is diff small?

for → when you know how many statements times you want to iterate. use counter.

while
↓
event based. we don't know how many times it'll iterate. wait for an event to ext the loop.

```
for index = expr
    statement 1
    statement 2
    statement n
end
```

counter

← typically first : incr : last

} body of loop

line it all up nicely.

ex.

```
for ii = 1:2:10
    disp(ii);
end
```

displays

1
3
5
7
9

always a scalar in this example

ex.

for v = [1 3; 4 5]

Never mess w/ v inside the loop like this →

```
disp('---------');
disp(v);
v = v, * 2;
disp('changed to');
disp(v);
end
```

displays →

```
---
1
4
Changed to
2
8
---
3
5
Changed to
6
10
```

while loops

```
while expr
    statement1
    statement2
    statement n
end
```

} body of loop

```
load econ_data.mat
x = [10;10;10;10]   % initial estimate
error = 1;          % initialization
while (abs(error) > 10^-5)
    old_x = x;
    x = C*x + d;
    error = old_x - x;
end
disp('The production vector is:');
disp(x);
```

if we picked d = x, and make error =

```
old_x = zeros(size(x));
thresh = input(...);
while (abs(old_x - x) > thresh)
    old_x = x
    x = C*x + d
end
```

if you put iterations = 500, then you can change while to obey the amount of iterations

```
while (abs(old_x - x) > thresh & iterations < 500
    old_x = x;
    x = C*x + d
```

↑ counter

```
x = c*x'+d
iterations = iteration+1
end
```

this is a counter. Value doesn't change magically. You need to update x.

A break statement stops the loop completely.

```
anyguess = ceil(6*rand(1));
while 1          % 1 means true, & it'll never change.
    x = input('Guess a number: ');
    if (x==myguess)
        disp('You found it!');
        break;          % means
    else
        disp('Sorry, that's not it.');
    end
end
```

A continue goes to the end of the statement, but goes back through the loop again.

Never test for equality between floating point #s.

```
x = .1;
while (x ~= 1)
    disp(x);
    x = x + .1;
end
```

displays →

0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
⋮

eek! It'll go forever!

Wasteful

```
ii = 1:5
data(ii) = some_value
end
```

better

```
data = zeros(5);
for ii = 1:5
data(ii) = some_value
end
```

There is an error in the slides on logical indexing.
If A is an array and B a logical array of the same

size, then A(B) returns the values of A whose corresponding index in B is 1. The values are returned in COLUMN major format, not in row major format as the lecture notes and our class example implied. For example:

A = [10 7 20 ;
19 9 18]

If B = A>10 then B is

[0 0 1;
1 0 1]

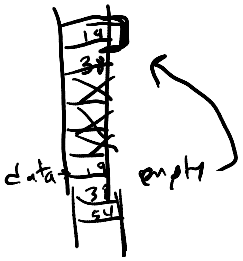
There are aces at indices 2, 5 and 6.

A(B) is [19;
20;
18]

Vana

Arrays, solving probs

Wednesday, October 04, 2006
6:55 AM



data(1) = 19
data(2) = 38
data(3) = 54

We should avoid enlarging an array inside the loop. To avoid this, tell it you need 1x1000 array.

ex.
```
tic;                    sets stopwatch
 for k = 1:500
  for m = 1:500
     data(k,m) = 1;
  end
end
elapsed_time = toc;   tells us how much time has passed
fprintf('Elapsed time with no preallocation: %6.4f seconds\n', elapsed_time);
clear all;
tic;
 data = zeros(500);
 etc
```

```
1 x10000
for k = 1:10,000
  cv(k) = cos(v(k))
end
```
takes too much time

cos(v) in beginning is better.

preallocating is always better.

index arrays

```
arr(1:3)
arr([1 5 9])
arr(2:end, 1:end)
```

Find out what elements of an array satisfies some property.

```
 1  0  -2
 6  -4  -5
```

A>0

```
 1  0  0   ← logical array
 1  0  0
```

A(B) will extract the elements that gave you a true (1)

```
 1
 6
```

And if it were
```
 1  0  0
 1  1  0
```
then you get [1 6 4]

B = A>0

```
 1  0  0
 1  0  0
```

You can determine if any elements exist that satisfy the property.

```
 1  -2  4
-1   3  9
```

A<0

```
 0  1  0
 1  0  0
```

A(negs)

```
 -2
 -1
```

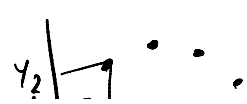
negs = A<0
if any(A(negs))
then...

A ball is thrown from position (0,0) w/ $V_0 = 20$ m/s at $\theta = 0°$. Write program that plots trajectories for the ball for all θs from 5°–85° in 10° steps. Program should also determine horizontal distance traveled by ball before it reaches the ground again for all θs from 0° to 85°, in 10° increments. Also determine θ that maximizes range of the ball, & plot that trajectory in a diff color w/ a thicker line

top → down
evaluate ranges
plot ranges → determining θ for max range.
evaluate trajectories
plot trajectories

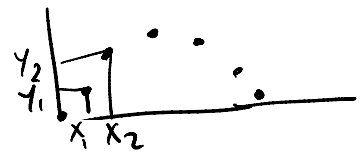
You don't write the whole program in one go.

$V_0 = 20$ m/s
$V_{0x} = V_0 \cos\theta$ ← velocity



$V_0 = 20\,m/s$

$V_x = V_{x0} * \cos\theta$
$V_y = V_{y0} * \sin\theta$ } velocity



$y(t) = y_0 + V_y t + \frac{1}{2} g t^2$ $\quad y_0 = 0$ at this problem

$y(t) = V_y t + \frac{1}{2} g t^2$

$x(t) = x_0 + V_x t$ $\quad x_0 = 0$ at this problem

$x(t) = V_x t$

radians = degrees *(pi /180)

We want to calculate the ranges (the x's) from 0°-90°

0 1 2 . . . 90

$x(t) = v_x t$

we know $0 = 0 + V_y > t + \frac{1}{2} * g * t^{\wedge}2$

$V_y = V_{y0} * \sin\theta$

$V_y = 2 * \sin\theta$

$t = \frac{-2V_y}{g}$ $\longrightarrow$ $V_x = 2 * \cos\theta$

$x = V_x * t$

In Matlab-English hybrid:

```
g = 9.81
v0 = 20
angles = 0 -90
Vy = v0*sin(angles. *ratio)
Vx = v0*cos(angles. *ratio)
```

Ranges for all angles (sample prob)

Thursday, October 05, 2006
7:00 AM

Back to yesterday's problem: Find ranges for all ∡s 0 - 90°

```
angles = 0:90;
rads = angles .*(pi/100);%converts to degrees

v0_x = v0 .*cos(rads);
v0_y = v0 .* sin(rads)

time_to_land = (-2/g) .* v0_y;
range = v0_x .* time_to_land;
```

Now we want a table:

angle	range
0	0.76 8.4f
1	—
2	—
⋮	
90	—

So:

```
fprintf('\n Angle      Max range \n');
for (k = 1:91)
  fprintf('%2d                %8.4f\n', angles(k), range(k));
```

(8 spaces) (4 decimal digits) f is for floating point

%2d ← integer

Now we want to find & print angle for max range
Because we set up angle(0) = range(1), angle(1) = range(2)
We can find the ∡.

```
[maxrange index] = max(range);
fprintf('Max range %6.2 fm at %d degrees\n\n')
```

Now we want trajectory
For each ∡, find out where it was at time t
So we use a for loop.

```
for angle = 5:10:85
  rad = angle * pi/180;
  v0_x ---
  v0_y ---
  time_to_land. —
  time_interval = time_to_land/99;
  time = 0:time_interval:time_to_land
  current_x = v0_x * time
```

% How long will it take to land?
% split in intervals - for each moment in time, find location of the ball. save to a table

if ∡ → max range → end

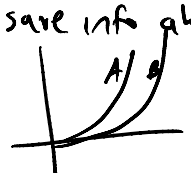
current_x = v0_x. *times; if ∠ → max range → etc
current_y = v0_y. *times + (.5*g).*(t then plot separately else plot normally plot(x, y, 'color, linewidth'
if angle == maxindex-1 plot
    plot(current_x, current_y, 'Linewidth', ...
else
    plot(current_x, current_y, '- - -
end
hold on; % we want to plot everything on the same figure, so if you put that there, it'll start in a fig 3 window next time. You don't want to lose the previous trajectory. % if it's 45 plot ~~end plot~~ else plot end

end
hold off;

A double every 60 min BACTERIA

B double every 90 min

Examine how they double, save info about their #s every 3 hours. Plot growth.



x: 0 1 2 3 4 5 6 ... 24
y_a y.1 y1 y2 y3
y_B y: 1 populations at x times (hrs)

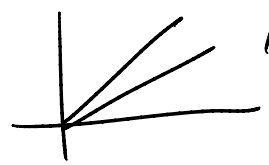
for 0 min ... ? 24*60 min
    notebook if it's a multiple of 60 min, then A reproduces.
    else if it's a multiple of 90 min, then B reproduces.
    (b/c mult of 60 could be mult. of 90)
    else if it's a multiple of 3 hours then save populations in our table.
end
plot.

rem( ) remainder function == 0
they double population.
check every 30 min, = increment.

numA = 1;
numB = 1;

```
for ticker = 30:30:total_min
    if rem(ticker,60) ==0
        numA = numA*2;
    end
    if rem(ticker,90) ==0
        numB = numB*2;
    end
    if rem(ticker,180) ==0
      population(1, index) = numA;
      population(2, index) = numB;
      index = index+1;
    end
figure(1);
plot(1:9, population(1,:));
hold on;
```

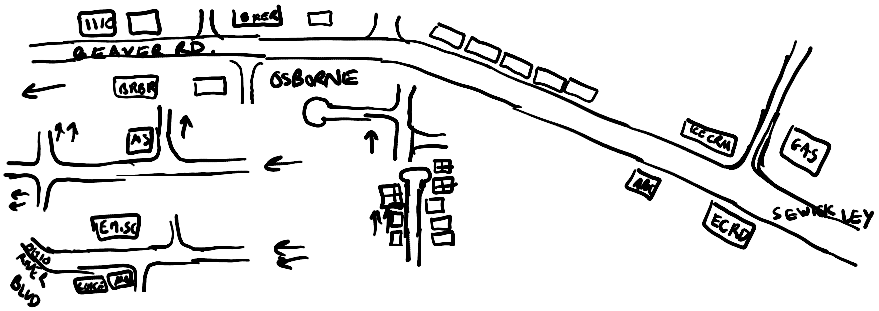
← loop - goes to next available position



log plot, not normal b/c you can't see it otherwise.

Exam review, Matlab Linear Algebra

Friday, October 06, 2006
6:58 AM



Exam covers everything up to & not including reduction. Located in Tech Auditorium

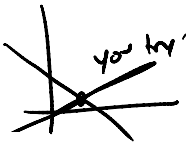
Matlab:

Solve $Ax = b$

Method 1: $x = A \backslash b$ → Gaussian Elimination (Matlab uses this)

Method 2: $inv(A) * b$ > not equivalent in Matlab's eyes.

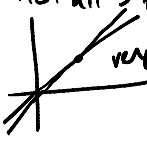
$x - 2y = 0$
$x + y = 1$
$y = 1 - x$
$y = \frac{x}{2}$



you try to see if they intersect.

but if they were ∥ then ∅, & if they were the same line, there are ∞ solutions.

3x3 systems intersect. Planes coinciding, being ∥, or 3 planes intersect but not all 3 form a pt.



very close together, even though it has a unique solution, but that will appear that it has an ∞ # of solns. ~~so~~ B/c we use floating pt. #s.

pretend that's the graph for $x - 2y = 0$, $x + y = 1$

$-x + 2y = 0$
$x + y = 1$
$3y = 1$
$y = \frac{1}{3}$

By adding them, we're moving them, rotating them without leaving the intersection point.

You multiply everything by a nonzero #.

FBI

$x + 0y + 0z = x_1$
$0x + y + 0z = y_1$
$0x + 0y + z = z_1$

Solving systems of linear equation

$2x + 3y + z = 10$
$-x + z = -1$
$3x - y + 2z = -7$

Start w/ augmented matrix:

$$\begin{bmatrix} 2 & 3 & 1 & 10 \\ -1 & 0 & 1 & -1 \\ 3 & -1 & 2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

leading values

echelon matrix

pivot

$$\begin{bmatrix} 1 & 0 & 0 & s1 \\ 0 & 1 & 0 & s2 \\ 0 & 0 & 1 & s3 \end{bmatrix}$$

Rules

① If there are any rows of all 0's, they

LLW01S3.1

Once you have the eschelon form, you can do a few more things like multiplying & adding to make #s 0.

① If there are any rows of all 0's, they will be at the bottom.
② Everything in the column below a leading value (1st * after 0) is 0
③ Every leading # is somewhere to the right of the leading # above it. * 0 * to the right of the last *.

$$\begin{bmatrix} 2 & 3 & 1 & 10 \\ -1 & 0 & 1 & -1 \\ 3 & -1 & 2 & -7 \end{bmatrix} \longrightarrow [(\text{row}2)3] + \text{row}3$$

$$\begin{bmatrix} 2 & 3 & 1 & 10 \\ -3 & 0 & 3 & -3 \\ 3 & -1 & 2 & -7 \end{bmatrix}$$ intermediate step

$$\begin{bmatrix} 2 & 3 & 1 & 10 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 5 & -10 \end{bmatrix} \longrightarrow [(\text{row}2)2] + \text{row}1$$ and you get this.

$$\begin{bmatrix} 2 & 3 & 1 & 10 \\ -2 & 0 & 2 & -2 \\ 0 & -1 & 5 & -10 \end{bmatrix}$$ intermediate

$$\begin{bmatrix} 0 & 3 & 3 & 8 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 5 & -10 \end{bmatrix}$$ and you get this. swap rows 1 + 2

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 3 & 3 & 8 \\ 0 & -1 & 5 & -10 \end{bmatrix} \longrightarrow [(\text{row}3)3] + \text{row}2$$

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 3 & 3 & 8 \\ 0 & -3 & 15 & -30 \end{bmatrix}$$ int.

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 0 & 18 & -22 \\ 0 & -1 & 5 & -10 \end{bmatrix} \longrightarrow$$ swap rows 2+3

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 5 & -10 \\ 0 & 0 & 18 & -22 \end{bmatrix} \longrightarrow$$ divide row 3 by 18

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 5 & -10 \\ 0 & 0 & 1 & -11/9 \end{bmatrix} \longrightarrow [(\text{row}3)-5] + \text{row}2$$

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 0 & -10 + \frac{55}{9} \\ 0 & 0 & 1 & -11/9 \end{bmatrix} \longrightarrow (\text{row}2)(-1)$$

$$\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 10 - \frac{55}{9} \\ 0 & 0 & 1 & -11/9 \end{bmatrix} \longrightarrow [(\text{row}3)(-1)] + \text{row}1$$

$$\begin{bmatrix} 1 & 0 & 0 & 11/9 - 1 \\ 0 & 1 & 0 & 10 - 55/9 \\ 0 & 0 & 1 & -11/9 \end{bmatrix}$$ and those are the answers, x1, y1, z1.

$$\begin{bmatrix} 1 & & & \\ 0 & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

0 = 0 is a true statement,
One free variable that can take any value, *.

Multiplying Matrices

Tuesday, October 10, 2006
7:04 AM

$Ax = b$

$x = A^{-1} * b$ Not a good way. Like doing this: $7x = 21$, $x = 7^{-1} \cdot 21$

instead, do this: $x = 21/7$. $x = b/A = A \backslash b$

3 types of operations you can apply to the rows of an augmented matrix
- Interchange

$[A \quad B]$ → $[I \quad b^*]$ is the solution

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} c1 \\ c2 \\ c3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} s1 \\ s2 \\ s3 \end{bmatrix}$$

When doing this by hand, you can swap rows. But when writing a program, you can't have it do random ops, hope for the best. You need a sequence

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$$

Mult. 1st row by $-a_{21}/a_{11}$ & add to 2nd row. Adds a 0 in position (2, 1)

$$a_{11}\left(-\frac{a_{21}}{a_{11}}\right) = -a_{21}$$
$$-a_{21} + a_{21} = 0$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ 0 & \left(a_{22} + a_{12}\left(-\frac{a_{21}}{a_{11}}\right)\right) & \left(a_{23} + a_{13}\left(-\frac{a_{21}}{a_{11}}\right)\right) \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

This is a <u>for</u> loop

Mult. 1st row by $-\frac{a_{31}}{a_{11}}$ & add to a_{31}

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} c1 \\ c2 \\ c3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 0 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -10 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 & 8 \\ 0 & 5 & 2 & -15 \\ 0 & 10 & 4 & -39 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 & 8 \\ 0 & 5 & 2 & -15 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

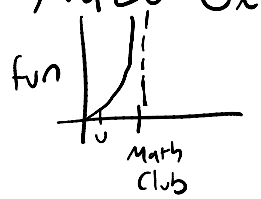
this says $0x + 0y + 0z = -9$
this means $\emptyset$.

When you have $0x + 0y + 0z = 0$,

then you only have 2 equations & 3 unknowns left. So you chose the variable whose fault it is, and you write the other variables in terms of the unknown.

This gives you an infinite amount of solutions.

Math Club



As $v \to$ Math club, fun $\to \infty$.

$$.00001x + 2y = 2$$
$$x + y = 2$$

$$\begin{bmatrix} .00001 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} .00001 & 2 & 2 \\ 0 & 1.99999 & 1.99998 \end{bmatrix} \to \begin{bmatrix} .00001 & 2 & 2 \\ 0 & 1 & .99999 \end{bmatrix}$$

$$\to \begin{bmatrix} .00001 & 0 & 0^* \\ 0 & & .99999 \end{bmatrix}$$

$0^* \approx 0$.
approx. soln $x = 0, y = 1$.
Incorrect.

Rounding errors cause the wrong answer.

Swap instead

$$\begin{bmatrix} .00001 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ .00001 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1.99999 & 1.99998 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & .99999 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1^* \\ 0 & 1 & .99999 \end{bmatrix}$$

$y^* = 1 \quad x^* = 1$

Matrix Addition

Wednesday, October 11, 2006
7:01 AM

Manufacturing
Agriculture
Services

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$x = \underline{Cx + d}$

1 unit M consumes .1 units M
1 unit A consumes .6 units M

x_2 units of A consumes $.6x_2$ M
x_1 units of M consumes $.1x_1$ M
x_3 units of S consumes $.6x_3$ M

$x_1 = .1x_1 + .6x_2 + .6x_3 + d_1$

same thing as

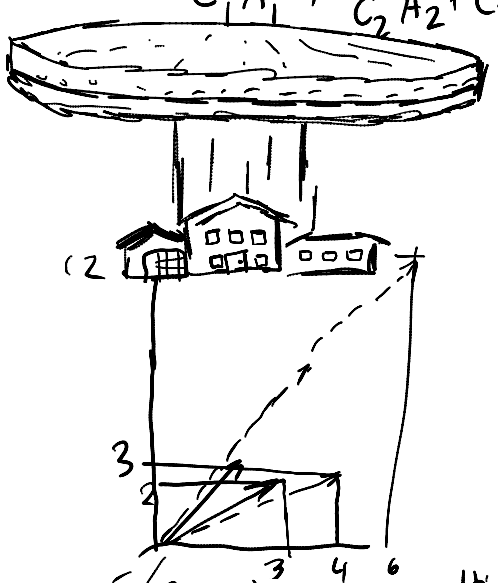
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + d_1 = x_1$$

another way to view the matrix is

$$\underset{A1}{\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}} * x_1 + \overset{A2}{\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}} * x_2 + \overset{A3}{\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}} * x_3 + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

linear combination of A1, A2, & A3 (column vectors)

$$c_1 A_1 + c_2 A_2 + c_3 A_3$$



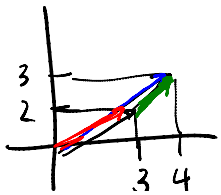
$x = Cx + d$

so 3 diff ways to solve.
Linear combination
Matrix
Linear system of equations

every time you multiply a vector by a scalar, you move its end pt. along the original line.

$$-5\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -15 \\ -30 \end{bmatrix} \qquad -1\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \end{bmatrix} \qquad 2\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

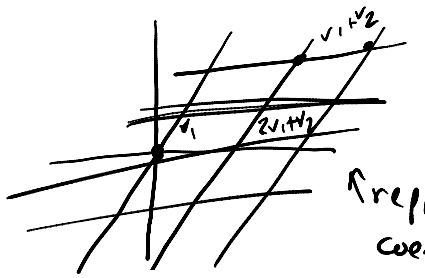
$$\begin{bmatrix}3\\2\end{bmatrix} + 1 = \begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$



$$\begin{bmatrix}3\\2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$$

$c(v_1 + v_2 + v_3 + v_4)$ or $av_1 + bv_2 + cv_3 + dv_4$

$2v_1 + v_2$



you have a grid when you add them together

↑ represents linear combination of vectors w/ some coefficients

$\underset{n\times 1}{v_1} \quad v_2 \quad v_3 \ldots v_p$ $\qquad c_1v_1 + c_2v_2 + c_3v_3 + \ldots + c_pv_p$

span, subset of R^n

does w belong in this span?

span $\{v_1, v_2 \ldots v_p\}$ = grid above.

$x_1v_1 + x_2v_2 = w$

$x_1\begin{bmatrix}2\\1\end{bmatrix} + x_2\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix}$

$v_1 = \begin{bmatrix}1\\-1\\-2\end{bmatrix} \quad v_2 = \begin{bmatrix}5\\-4\\-7\end{bmatrix}$

$v_3 = \begin{bmatrix}-3\\1\\0\end{bmatrix} \quad y = \begin{bmatrix}-4\\3\\h\end{bmatrix}$

we want to find h so that y is in the span v_1 v_2 v_3

$$\begin{bmatrix}1\\-1\\-2\end{bmatrix}x_1 + \begin{bmatrix}5\\-4\\-7\end{bmatrix}x_2 + \begin{bmatrix}-3\\1\\0\end{bmatrix}x_3 = \begin{bmatrix}-4\\3\\h\end{bmatrix}$$

$$\begin{bmatrix}1 & 5 & -3\\-1 & -4 & 1\\-2 & -7 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-4\\3\\h\end{bmatrix}$$

$$\begin{bmatrix}1 & 5 & -3\\0 & 1 & -2\\-2 & -7 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-4\\-1\\-8+h\end{bmatrix}$$

$$\begin{bmatrix}1 & 5 & -3\\0 & 1 & -2\\0 & 3 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} =$$

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -5+h \end{bmatrix}$$

$$0 = -5 + h$$
$$5 = h$$

If h is not 5, the system will not have a solution so y is not in the span. As long as $h=5$, y <u>is</u> in the span

Let A be an $m \times n$ matrix.

- Columns of "A" span in the augmented matrix
- $Ax=b$ has a solution for every b in the augmented matrix
- Every b in the augmented matrix is a linear combo. of the columns of A
- A has a pivot position in every row

A is the coefficient matrix, not the augmented.

If one of these is true, the others are. If one is not true, the others aren't.

If you want to prove something is not true, then the others will be not true as well.

Could a set of 3 vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$?

$$\begin{array}{c} \begin{bmatrix} + & - & - \\ - & + & - \\ - & 0 & + \\ 0 & 0 & 0 \end{bmatrix} \\ v_3 \end{array} \begin{array}{c} \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} \leftarrow b \end{array}$$

So no, b/c every ~~position~~ row does not have a pivot position.

4x5 matrix: is it possible to delete a column & still have the remaining ones span $\mathbb{R}^4$?

$$\begin{array}{c} \begin{bmatrix} 1 & - & - & - & - \\ 0 & 1 & - & - & - \\ 0 & 0 & 1 & - & - \\ 0 & 0 & 0 & 1 & - \end{bmatrix} \\ v_3 \end{array}$$

While Loops

$$x_{n+1} = C x_n + d$$
$$x_1 = C x_0 + d$$
$$x_2 = C x_1 + d$$
$$x_3 = C x_2 + d \qquad \text{so} \quad x_n = C x_n + d$$

Sum

Thursday, October 12, 2006
7:01 AM

$$\frac{1}{x^0} + \frac{1}{x^1} + \frac{1}{x^2} + \frac{1}{x^3} + \dots + \frac{1}{x^n}$$

sum = 0
sum = sum + $\frac{1}{x^1}$
sum = sum + $\frac{1}{x^2}$
⋮

```
sum=sum+1/x^k
sum = 0
for is used when we know how many times we want to do
for index=0:12          index=counter
    sum=sum + 1/x^index
end

for ii=1:2:5            ii=1, ii=3, ii=5
    A(ii) = x           % the 1st, 3rd, + 5th element of array A will = x.
end


while (condition)       % the moment condition ≠ true, it stops.
    =                   Condition needs to change. It needs
    =                   to become false or the loop will continue
end                     forever.

sum=0
x=1
while x>0               % This will go on forever
    sum=sum + 1/x;
end
```

$$0 \neq \frac{1}{1} = \frac{1}{1}$$
$$\frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

```
sum=0
x=1
while x>0
    sum=sum+ 1/x;
    if x==1 break;      % this would execute only once
end
end
sum=y
sum=0
x=1
while x>1
    sum=sum+ 1/x;
```

```
sum=0
x=1
while x>1
   sum=sum+ 1/x;
   if x==1 break;
end end
```

```
go on = input('Continue? Y/N': )
```

```
A=[            ]
[m n] = size (A);
 for r= 1:m
    for c= 1:n
     A=(r, c)=0
    end
 end
```

```
r=1
   c=1
     A(1,1)=0
   c=2
     A(1,2)=0
   c=3

   c=n
     A(1,n)=0
r=2
   c=1
     A(2,1)=0
   c=2
     A(3,1)=0
   etc.
```

% This is inefficient because you're accessing the matrix row by row. In Matlab, matrices are stored in column format. You're jumping around in memory.

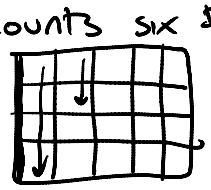
% You could have eliminated the inner loop by using a colon.

```
For r=1:m
   A(r, :) = 0
end
```

A([2 4], :) 2nd + 4th row completely.

A([2 2], :) Access the 2nd row twice

m×n matrix A(6) Counts six #'s in each column



ones(5)' 5×5 matrix of all 1's

ones(5,4)

eye(size(B)) [illegible]

eye(size(B)) identity matrix of size that's been specified

A(1:2:5, :) = B(1:2:5, :)

it will copy rows 1, 2, & 5 from A completely into B.

only if size(A) = size(B)

A(1:2:5, 4:end) 4 to the last colum, 1st, 3rd, & 5 elements in the row. So (1,4) to end of colum. (3,4) to end of its colum.

A = zeros(5)

A([1 3], [2 5]) = [0 1; 1 2]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A = zeros(4,3); A(:,1) = 1:2:5; x = A(4,3)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Sizes do not compute.

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$ There is no 4th row now.

break takes you out of a loop

continue skips the iteration but goes back to the top

return terminates the program outside of the loop

If you're doing an array operation, you need the dot.

Every time you have a x=%6.2f, y = %d

This prints a # that takes up 6 spaces, 2 of which are after the decimal point, and then %d, an integer.

Matrices

Friday, October 13, 2006
7:00 AM

---

1. Arithmetic	1. Negations	~1 | 0 negation 1st
2. Relational	2. Arithmetic	not
3. Negations	3. Relational	0 | ~1
what Chapman says	The right way	

If a plane is spanned by vectors $\vec{v}$ & $\vec{u}$, then every vector $\vec{b}$ located in this plane can be expressed as a linear combination of $\vec{v}$ & $\vec{u}$

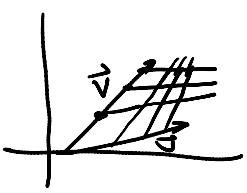
$$b = x_1\vec{v} + x_2\vec{u}$$

$$b = x_1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + x_2\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$b = \begin{bmatrix} v_1 & u_1 \\ v_2 & u_2 \\ v_3 & u_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If you want to know whether a certain vector lies in the hyperplane spanned by $\vec{v}$ & $\vec{u}$, find coefficients x_1 & x_2 so the system

$$\begin{bmatrix} v_1 & u_1 \\ v_2 & u_2 \\ v_3 & u_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



More practical: Open Leontif model if the econ probs we have

1 Closed Leontif model. Small alien civilization completely self sufficient. Nothing comes from outside, nothing goes out.

The farming alien consumes 40% of the food, 31% of the clothes, 44% of the energy.

The tailor alien consumes 25% of the food, 36% of the clothes, & 20% of the energy.

The coal mining alien consumes 35% of the food 33% of the clothes, & 36% of the energy

What should be the price of food, clothes, & energy so that the income of each person exactly equals the output? You want everything produced to be consumed.

$.40f + .31c + .44e = f$
$.25f + .36c + .20e = c$
$.35f + .33c + .36e = e$

$-.60f + .31c + .44e = 0$
$.25f + -.64c + .20e = 0$
$.35f + .33c + -.64e = 0$

$$\begin{bmatrix} -.60 & .31 & .44 \\ .25 & -.64 & .20 \\ .35 & .33 & .64 \end{bmatrix}\begin{bmatrix} f \\ c \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

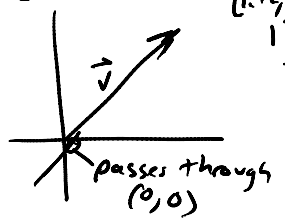
reduce matrix

$$\begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} f \\ c \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Cx = x \nearrow \begin{bmatrix} f \\ c \\ e \end{bmatrix}$
$Cx - Ix = 0$
$(C-I)x = 0$

Homogeneous systems when right side $= 0$.
system has a trivial soln, 0. But iff the equation has ~~one~~ at least 1 free variable, we'll have a nontrivial soln.

$\vec{x} = e \cdot \vec{v}$ where $\vec{v} = [1.12, .75, 1]$



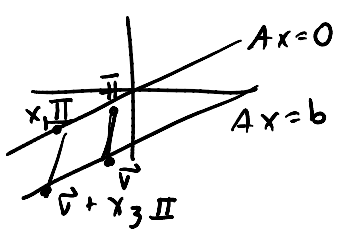
$\vec{x} = \text{span}(\vec{v})$
$x = e^* v$ // represents a line
$\vec{x} = c_1\vec{v} + c_2\vec{v}$ // plane

$$\begin{bmatrix} 1 & 0 & 1.6 \\ 0 & 1 & .4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}$$

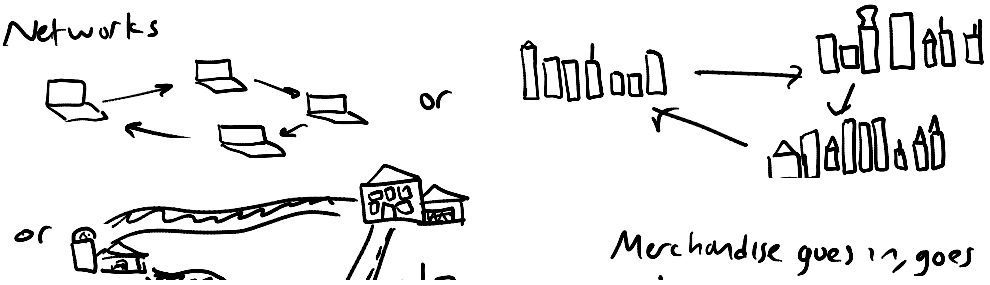
$x_1 + 1.6x_3 = -4$
$x_2 + .4x_2 = -3$
$x_3 + s$ free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} -4 - 1.6x_1 \\ -3 - .4x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}\begin{bmatrix} -1.6x_1 \\ -.4x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1.6 \\ -.4 \\ 1 \end{bmatrix}$$

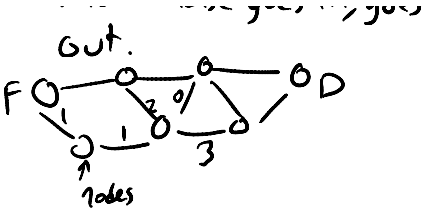
?

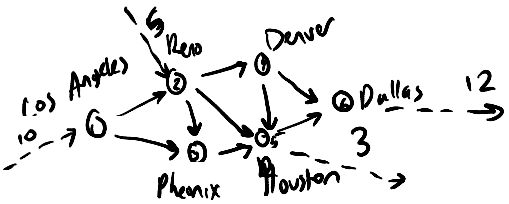


Networks



Merchandise goes in, goes





TEN-THIRTEEN

$\boxed{1013}$

x_{13}
x_{12}
x_{23}
x_{24}
x_{25}
x_{35}
x_{45}
x_{46}
x_{56}

$x_{13} + x_{23} = x_{35}$
$x_{24} = x_{45} + x_{46}$
$x_{46} + x_{56} = 12$
$x_{23} + x_{35} + x_{45} = x_{56} + 3$
$x_{12} + 5 = x_{23} + x_{24} + x_{25}$
$10 = x_{12} + x_{13}$

They're gray!

So we build an order. Pick an equation.

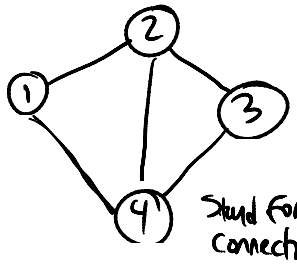
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{12} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{35} \\ x_{45} \\ x_{46} \\ x_{56} \end{bmatrix} = \begin{bmatrix} 3 \\ \\ \\ \\ \end{bmatrix}$$

$x_{23} + x_{35} + x_{45} - x_{56} = 3$

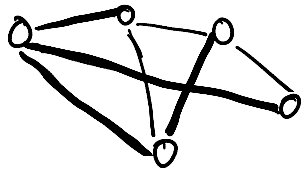
$$\begin{bmatrix} \text{Really huge matrix} \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 3 \\ -5 \\ -10 \end{bmatrix}$$

Spanning Tree

Tuesday, October 17, 2006
7:06 AM



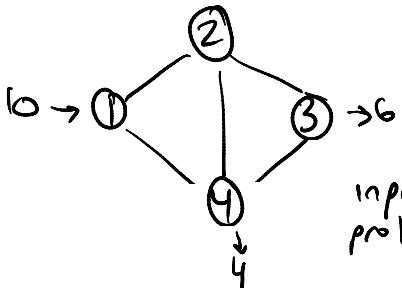
Stand for connections



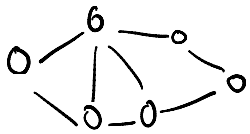
distance

Stand for roads ↑
Which roads should we pave?

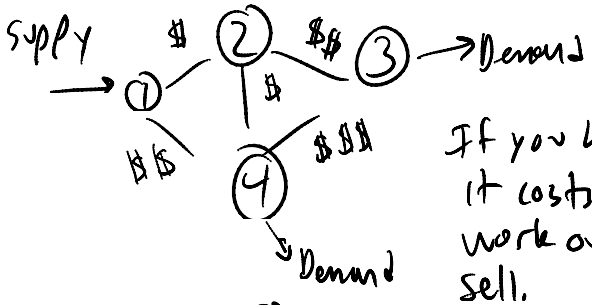
This forms a <u>spanning tree</u> - a graph that has no cycles.



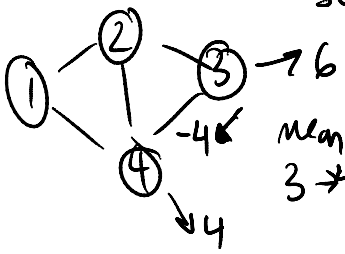
input-output problem



Max. capacity at each node, find max. amount of product you can generate utilizing as many of the routes & capacities as often.



If you have information on how much it costs to go through each loop, you can work out how much it costs to transport & sell.



Meaning you can only go from 4→3 not 3→4. But not in this problem

① $x_{12} + x_{14} = 10$
② $x_{24} + x_{23} = x_{12}$
④ $x_{43} + x_{42} = x_{14}$
③ $x_{34} + x_{32} = 6$

x_{12}	x_{14}	x_{24}	x_{23}	x_{42}	x_{43}	x_{34}	
1	1	0	0	0	0		10
0	0		1				

③ $x_{23} + x_{43} = 6$

⑤ $x_{14} + x_{34} + x_{24} = 4$

0 0 1

so you continue this. If you have negative flow, you just subtract that from the appropriate equation.

After you find parametric trans,

$6 + x_{24} - x_{43} = x_{12}$

$9 - x_{24} + x_{43} = x_{14}$

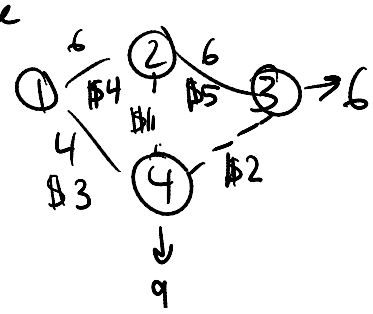
$6 - x_{43} = x_{23}$

x_{24} = free

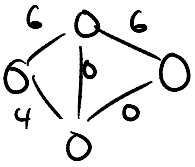
x_{43} = free

basic soln

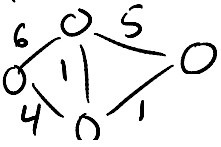
$$\begin{bmatrix} x_{12} \\ x_{14} \\ x_{23} \\ x_{24} \\ x_{43} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_{24} + \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_{43}$$



So if you have info about the money, you want to take advantage of the cheaper soln.

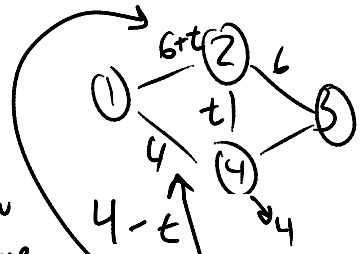


So now you don't want 6 from 2→3 you want



When you add/remove flow it affects other edges that form a cycle.

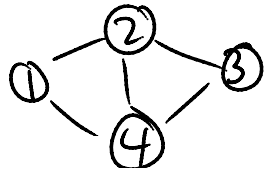
Now it's not a spanning tree ~~because~~ but it's an improved soln even if it has cycles.



examine how the flow along the cycle will change if you push more flow along the chosen edge.

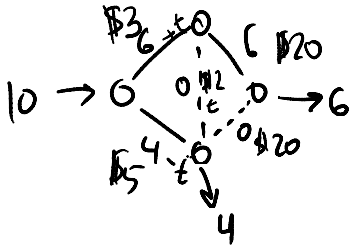
4 cannot change, so t would mean changes. And t = 0

Add it up and see if you lose or gain money.



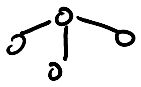
MATLAB coding

Wednesday, October 18, 2006
7:03 AM



The basic soln is a spanning tree
If you add the middle edge, you get a cycle.

$\$2(6+t) + 3(6+t) + 5(4-t) = \$\$$ If you end up w/ less \$\$, then it's good - you saved \$. If more \$\$, don't do it. It's a waste of time & money.

If you subtract enough from one branch, you get a new soln w/out that branch included.  In the end, you should end up w/ a spanning tree. If you move the columns around, you end up w/ a diff spanning tree.

---

if _____
sort( )

Sort being a function you come upon.

if _____
sort( )

A = [ ]

If you loaded this up and you previously had an A, it would override the previous A. Same is not true for functions. What happens in a function stays in a function. Like Vegas. Or Risa.

WHAT HAPPENS ON
RISA
STAYS ON RISA

WHAT HAPPENS IN
f(x)
STAYS IN f(x)

WRITE THE WORD "FUNCTION"

```
name.m
function[outarg1, outarg2 ...] =name[inarg1...]
% Name description
% Help text
% Comments
Statements
```

T(x)
STAYS IN F(x)

sounds like a good t-shirt,

outarg1, outarg2 are placeholders. They hold the final results in your program.

function [N, M] = size(A)

(rows, columns) = size(B)

```
>>help size
>>size
>> look for average
```

help lines with no blank lines in between
No
% help text

It will see this as the beginning of the function.

% Comments
Statements

There are a lot of comments that go into this type of work.

```
temp = (temp-32) * 5/9        % Convert F to C.
                                NOT % temp-32 times 5/9.

[n,m] = size(A);
% verify it's a column vector
if m~=1
   error('Input must be a column vector')
end

% compute the average
 sum=0;
 for ii=1:n
    sum = sum + A(ii)
 end
 ave = sum/n;
```

```
avg = sum/n;
```

```
%          input arg - column vector
% beginning | avg - (output arg) avg. of values
```

parameter passing

```
v = [1; 2; 3; 4]
x = average(v)
```

plugs v into A, but doesn't actually change v.

MATLAB sorts

Thursday, October 19, 2006
7:03 AM

```
n=size(A)
[n m]=size(A)
```

rows columns

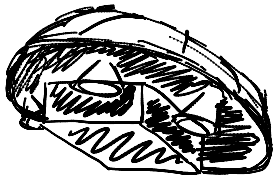
Sorting means taking a sequence of elmts. & ordering them
- #s from largest to smallest
- names from A-Z
- etc.

<u>Insertion sort</u> is not very good for big arrays, but if the array is small, or if the array is more or less sorted. You go along until you find one that needs to be behind/in front of current ones.

Another method is to split the pile into smaller piles and then even smaller ones. Until it's done.

<u>Bucket sort</u> is when you have lots of piles and you put each one in the right pile as you go.

<u>Selection sort</u> is the worst method. You go through the whole thing & find which one should be 1st. Then you find the one that should be done.



<u>Insertion sort</u>

20 | 78, 45, 5, 20
20 78 | 45, 5, 20
20 45 78 | 5, 20
5, 20, 45, 78 | 20
5, 20, 20, 45, 78 |

CAN WI GET PAST 17?

Start with wall at position 2 : #of elmts. in array.

el = element at the wall % to the right

Starting at (wall-1) move left as long as we haven't hit the edge & haven't found a place for el
Shift el to the right
repeat

A(index) = saved element

```
function [sorted] = isort(v, direction)
% isort sorts a vector using insertion sort in ascending or descending order.
% 'a' for ascending ...
% More info for user.
checks # args   % (verify correct #of args)
```

checks # args %(verify correct # of args) # of args specified

```
% nargchk(min # args, max # args, nargin)
message = narychk( 1, 2, nargin)
    error (message);
```

if there is no problem, error message will be empty string.

```
%.We set the direction to be 'a' by default. Has to be 'a' or 'd' or print error
% message.
```

CAN WE GET THROUGH 289?

```
if(nargin <2)
    direction = 'a';
end
if (direction ~= 'a' & direction ~= 'd')
  error('Invalid direction. Legal values...')
end

[rows cols] = size(v);
if~(rows ==1  or  cols ==1)
    error ('Input array should be a vector. You suck')
end
```

% if rows ~=1 & cols ~=1

% if(rows ==1 | cols ==1)

For loop

Friday, October 20, 2006
7:04 AM

1 or 2 input args
arg1 : row/col vector
arg2 : (opt.) 'a' 'd'
1 output : sorted vector
Algorithm = insertion sort

function[sorted] = isort (v, s)

this is different than
this

v = [5 2 1 6 3]
sortedv = isort(v)
sortedv2 = isort(v,'d')

msg = nargchk(1, 2, nargin)
1 to 2 inputs. If you wanted 1-4 inputs, range would be 1-4.

If # of args are valid, it will contain a blank line.
if # of args is wrong, it will print out an error message

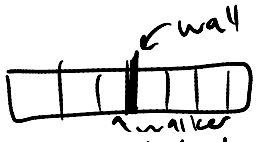
```
if nargin == 1,
    direction = 'a'
end
```

if the user doesn't specify, then the argument is automatically a.

```
if direction ~= a & direction ~= d
    error('Legal values are...')
end

[rows cols] = size(v)
if ~[rows == 1 | cols == 1]   % if rows ~= 1 or cols ~= 1
    error('Input should be row or column vector')
end
```

Repeating the operation - once you go through and insert elements, you can go through again. For loop.




```
for wall = 2 : length(v)
    element = v(wall);          % saving elmt by wall
    walker = wall - 1           % to check each value, we need a while loop
                                  b/c we don't know how far it will have
                                  to go.
    while (element < v(walker)  % elmt is less than item at the walker &
          && walker > 0)          walker hasn't hit the edge
        v(walker + 1) = v(walker)
        walker = walker - 1
    end
    v(walker + 1) = element     % b/c that's the last empty slot
end

Sorted = v                      % assigns v to the output argument so that will
                                  become the value of the function.
```

% but we're only sorting ascending

% && allows each thing to be evaluated separately. If one is false, so is the other.

% & only tests one at a time w/ no regard for the other.

System Consistency

Tuesday, October 24, 2006
7:09 AM

linear search
- Start at beginning
- keep checking
- if you find it
  - return success
- all the way to the end
- return failure

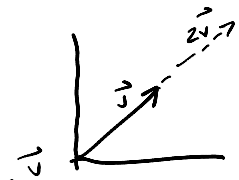
```
function[found] = linsearch(A,target)
    % LINSEARCH searches for target in matrix A
% help text
blank line

% comments
message=nargchk(2,2,nargin)
error(message);
found =0   % initially you haven't found it
[m n] = size(A)
numelmts = m*n
for ii = 1:numelmts;
    if target == A(ii)
        found=1
        break
    else
        continue
end
end
```

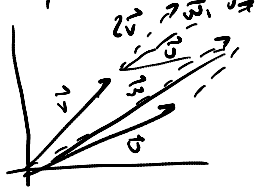
for loops nested is another option. One for loop to go through all cols, one to go through all rows. A(ii) tells you what elmt you're on.

% useless in this case.

% you can have found=0 on the bottom if we change break to return. Return stops execution of the function.



A single vector spans a line of the same slope. Multiply the vector by a scalar to get a new point



$\vec{u} + \vec{v} = \vec{w}$
$2\vec{v} + \vec{u} = \vec{w}_1$

Linear dependence of vectors

$x_3\vec{w} = x_1\vec{u} + x_2\vec{v} \rightarrow x_1\vec{u} + x_2\vec{v} - x_3\vec{w} = 0$

$\vec{w}$ is a linear combination of $\vec{u}$ & $\vec{v}$.

If at least one is a linear combination of another, you say the whole set of vectors is linearly dependent.

$c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_p\vec{v_p} = 0$

where one set of weights $\{c_1, c_2, \ldots c_p\}$ where weights $\neq 0$
at least

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ Are they linearly dependent?

$v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ See if the system is consistent

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} c \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

0,0,0 is a trivial soln. We're looking for a non-trivial soln.
We're hoping for a free variable

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1 + 2c_3 = 0$
$c_2 + c_3 = 0$
$0c_3 = 0$

$x_3\vec{w} = x_1\vec{u} + x_2\vec{v} \rightarrow x_1\vec{u} + x_2\vec{v} - x_3\vec{w} \neq 0$

For linear independency, you should expect a pivot in every column. You want exactly one soln. Independence means 0,0,0 is the only soln.

If you can find out of a collection of vectors, one of them is a linear combination of the others, then it is dependent.

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$
$\vec{u}$ $\vec{v}$ $\vec{w}$

$\vec{w} = \vec{v} + \vec{u}$
$1\vec{w} - 1\vec{v} - 1\vec{u} = 0$

To prove they're independent, none of them are a linear combo of others.

$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ yes, linear combo

$\vec{v}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\vec{u}$ $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ $\vec{w}$ $\begin{bmatrix} -1 \\ 8 \\ 3 \end{bmatrix}$

You can take anything times 0 vector.
$5\vec{u} + 0\vec{v} + 0\vec{w} = 0$
$\{c_1, c_2, c_3\} \neq 0$

A $m \times n$ for every $b \in \mathbb{R}^m$ $Ax = b$ has at most 1 soln
Why must the cols of A be linearly independent?

$Ax = b$ even if $b = 0$
$Ax = 0$, system has at most one soln. This has, at very least a trivial soln (all 0's). Therefore that soln has to be that

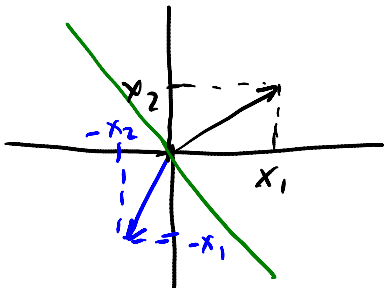
be that.

If coefficients x are all 0, cols of A are linearly independent by definition.

Matrix Transformations

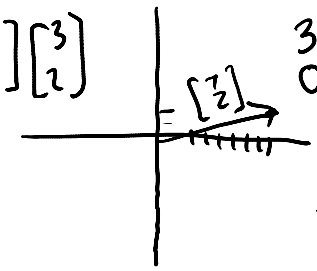
Wednesday, October 25, 2006
7:02 AM

$Ax = b$ we have a vector x, we apply a transformation A, & we view vector b.



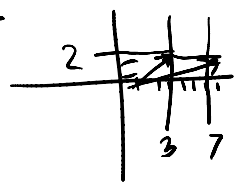
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



$3 + 4 = 7$
$0 + 2 = 2$

$$\begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

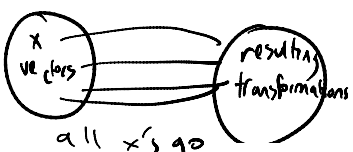


You pulled $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$

Because of the 0, the 2 stays the same.

Height doesn't change called a shear transformation

$\boxed{\text{SETAG CHOICES, INC.}}$



all x's go over...
infinite # of resulting transformations.
If no x's go over, no soln to set.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\mathbb{R}^3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \rightarrow \underset{\mathbb{R}^3}{\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}}$$

The vector was projected onto 2 axes only, so you get 2 things filled in, not 3.

$\underbrace{T(x)}_{Ax} = b$ Same thing, diff way to write.

$T(x) = Ax$ any vector in $\mathbb{R}^n$ Domain b = any vector in $\mathbb{R}^m$, codomain

$T(x)$ = "image of x" All images together = range. Not necessarily = to codomain, but that is a possibility

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ & $\vec{v}$ in domain of T

$T(k\vec{u}) = k\,T(\vec{u})$, k = scalar

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SACRIFICE

BY CONNOR O'MALLEY

$T(x) = Ax + b$ $b \neq 0$,

$T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) + b$

$= \underbrace{A\vec{u}}_{\text{not } T(\vec{u})} + \underbrace{A\vec{v} + b}_{T(\vec{v})}$

$T(\vec{u}) + T(\vec{v}) = A\vec{u} + b + A\vec{v} + b$ not the same as

$\therefore$ This particular transformation is not linear.

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2\right)$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2\right)$

$x_1\, T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2\, T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

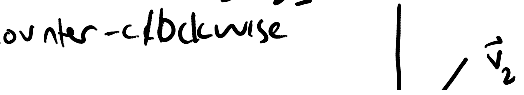
$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{pmatrix} 5 \\ -7 \end{pmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

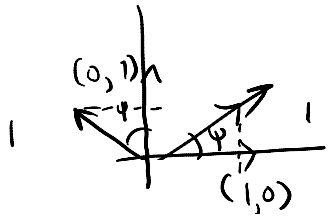
how does the transformation affect x_1, x_2? what is the standard transformation matrix A?

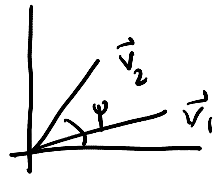
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 5 & 3 \\ -7 & 2 \end{bmatrix}}_{A}$

Standard matrix for the transformation that will rotate a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by $\measuredangle\ \psi$ (pos) counter-clockwise



counter-clockwise





$\measuredangle 1$

$\sin\psi = \frac{y}{1}$ $\quad \sin\psi = y$

$\cos\psi = \frac{x}{1}$ $\quad \cos\psi = x$

$\measuredangle 2$

$\sin\psi = \frac{-x}{1}$ $\quad \sin\psi = -x$

$\cos\psi = \frac{y}{1}$ $\quad \cos\psi = y$

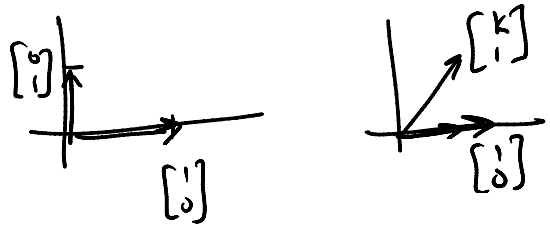
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\psi\\ \cos\psi\end{bmatrix}$$

$$A = \begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}$$

Horizontal shear from $\mathbb{R}^3$ to $\mathbb{R}^2$

$$\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}x_1 + kx_2\\ x_2\end{bmatrix}$$



$Ax = b$ $\quad$ You can have 2 vectors that will both give you b.
$Ay = b$

"Onto" Mapping means every vector in b is the image of at least 1 x. We cover all of $\mathbb{R}^m$, in other words, we span all of $\mathbb{R}^m$.

$\mathbb{R}^n \to \mathbb{R}^m$

Columns of A span $\mathbb{R}^m$.
which means $Ax = b$ always has a soln.
Each vector b in $\mathbb{R}^m$ is the image of at least one vector x in $\mathbb{R}^n$. Must be a pivot position in each row of A.

If one of Theorem 4 conditions are satisfied, then you can

say 'yes, it's $\mathbb{R}^2$ or whatever it is.

If it's true for every b, you can pick 0.

$Ax=0$ For every b, there is at most one x that matches it. They already have at least one soln if $A\neq 0$, and so this system has the trivial soln. This definition tells us there is exactly one soln, or in this case it's trivial.

Every column has a pivot, so columns are linearly independent.

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + x_3,\ x_2 - 6x_3)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 5x_2 + x_3 \\ x_2 - 6x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

one-to-one or onto ?

<u>Onto</u>

$\mathbb{R}^3$ onto $\mathbb{R}^2$ For every b there is an x that gives us the b. So the system has a soln.

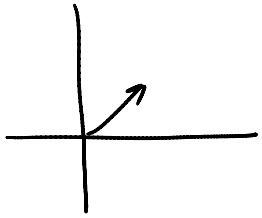
Every row has a pivot so soln is onto.

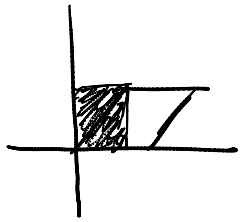
Not one-to-one

Transformations

Friday, October 27, 2006
7:28 AM

$Ax = b$





$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
$T(c\vec{u}) = c\,T(\vec{u})$

$T(\vec{u}) + T(\vec{v}) = 3\vec{u} + 3\vec{v}$
$T(c\vec{v}) = 3(c\vec{v}) = c(3\vec{v}) = c(T\vec{v})$

X-FILES

FOX MULDER

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 3|x_2| \end{bmatrix}$$

$x_2 < 0 \rightarrow -x_2$
$x_2 > 0 \rightarrow x_2$

$$c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

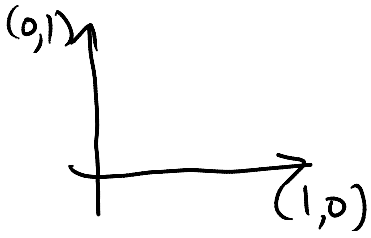
Has to be true for every scalar. So come up with one that disproves it.

$c = -1$

$$T\left(\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}\right) = \begin{bmatrix} 4(-x_1) - 2(-x_2) \\ 3|-x_2| \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ 3x_2 \end{bmatrix}$$

$$c\,T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = -1\begin{bmatrix} 4x_1 - 2x_2 \\ 3|x_2| \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ -3|x_2| \end{bmatrix}$$

So it doesn't work. Transformation isn't linear.



unit vectors

$T(x)$
$Ax = b$
$A\left[T(\underline{e_1}) \dots T(\underline{e_n})\right]$

columns of the identity matrix

$$x = Ix = [e_1\ e_2 \dots e_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$Tx = T(Ix) = T([e_1\ e_2 \dots e_n])$$
$$= T([e_1x_1 + e_2x_2 + \dots e_nx_n])$$
$$= T(e_1x_1) + T(e_2x_2) + T \dots (e_nx_n)$$
$$= T(e_1)x_1 + T(e_2)x_2 + T(e_n)x$$

X-FILES

$$= T(e_1)x_1 + T(e_2)x_2 + \dots + T(e_n)x_n$$

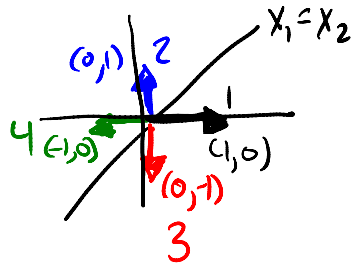
$$= \underbrace{[T(e_1)\,T(e_2)\cdots T(e_n)]}_{\text{Standard matrix of the transformation } A}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

DANA SCULLY

FOX MULDER

p. 90

8. $T: \mathbb{R}^2 \to \mathbb{R}^2$
through horizontal x_1
through line $x_1 = x_2$



Reflections go like this, final position is (-1,0)

Shear $e_2 \to e_2 + 3e_1$

$\mathbb{R}^2 \to \mathbb{R}^2$

$T(e_1) = e_1$

$T(e_2) = e_2 + 3e_1$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

general form is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ for vertical

Standard of Transformation

Tuesday, October 31, 2006
8:02 AM

for ii = 1 : size(
ii = jj
if jj = 1

for ii = 1 : size(B)

A(ii, col) if ii = 1, ii = jj
jj is row
col is column
A(col, ii)

for each column
find potential pivot
for row where pivot is
check whether it's preceded by all 0's 0 : col − 1 = 0
if yes,
pivot

A = [1 −3 2 ; 0 −1 4 ; 3 −5 −9]
Find image of [1 1 1]'

← standard matrix of transformation
$Ax = b$ ← domain
$T : \mathbb{R}^n \to \mathbb{R}^m$ ← codomain
$x \in \mathbb{R}^n$
$T(x) = b \in \mathbb{R}^m$ linear transformation

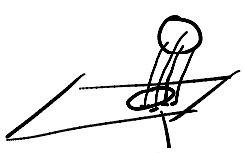


image is individual b
whole image collection = range

If you're given A & x (as in above), multiply A by x to get b.

$\mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} 1 & -5 & 7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & -7 \\ 0 & -8 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & -7 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-3x_3 \\ 1-2x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3$$

Does such an x exist?
Is this x unique?
Is every b the image of at most 1 x? In other words, are columns of A linearly independent? This would mean one-to-one.

If every b is the image of at least 1 x, it's onto.
If every b is the image of at most 1 x, it's one-to-one.

It's possible for a transformation to be both, if the transformation is the image of <u>exactly</u> one x, then it is onto & one-to-one.

If you can prove for $Ax=0$ that you have a nontrivial solution, you've found one b that breaks property. Transformation is not one-to-one.

$T: \mathbb{R}^n \to \mathbb{R}^m$
$A = m \times n$

$$\begin{bmatrix} x_1 & \cdots & x_n \\ \vdots & & \\ x_m & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n \quad n \times 1 \to m \times 1$

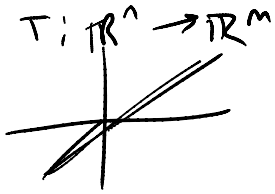
For every b, we have a solution in our system. That means n should equal the # of pivots

If we know something is one-to-one, we know that if there is a solution, it will be unique. n pivots

It's possible for m & n to be the same. If you know the transformation is onto (or one-to-one) do you know if the

transformation has to be $m \times n$?

$T: \mathbb{R}^n \to \mathbb{R}^m$



1. Reflect one way
2. Reflect other way

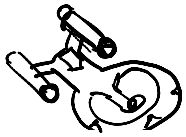
$T_1(x) = b \qquad A_1 x = b$

$T_2(b) = c \qquad A_2 b = c$

$A_2(A_1 x) = c$

$(A_2 \cdot A_1) x = c \qquad$ so $A = A_2 \cdot A_1$

$A = (A_k \cdot A_{k-1} \cdot \ldots \cdot A_1)$

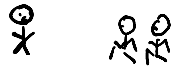


CAPTAIN JONATHAN ARCHER

Associative

Distributive

Not commutative $\qquad AB = AC$ does not necessarily mean $B = C$



$AB \neq BA$

$AB = 0$ does not necessarily mean $A = 0$ or $B = 0$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(cA)^T = cA^T$

$(AB)^T = B^T A^T$

Not every matrix can be inverted. For a matrix to be inverted, $(A^{-1})A = I_n$

$(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^{-1})^T = (A^T)^{-1}$

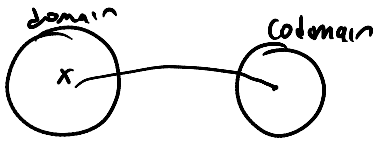
$(A^T)(A^T)^{-1} = I$

$(A^T)(A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I$

Domains, Codomains, Transformations

Wednesday, November 01, 2006
8:08 AM

onto $n >= m$
1-1 $m >= n$
onto & 1-1 $m = n$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ this gives you 1-1 & onto

$$T(x) = Bx = [b1 \ b2 \cdots bn] \begin{bmatrix} x1 \\ x2 \\ \vdots \\ xn \end{bmatrix} = b1x1 + b2x_2 + \cdots + bnxn$$

$$T2(T(x)) = C(Bx) = Cb1x1 + Cb2x2 + \cdots + Cbnxn$$

$$= C[b1 \ b2 \cdots bn] \begin{bmatrix} x1 \\ x2 \\ xn \end{bmatrix} = \underbrace{(CB)x}_{m \times 1 \text{ vector}}$$

$m \times p$ matrix

$p \times n$ matrix

$n \times 1$ vector

$xy = xz$
$\frac{1}{x} xy = \frac{1}{x} xz$
$y = z$

$xy = xz$ what if x is 0? $\frac{1}{x}$ doesn't exist. This breaks the rule.

$A^{-1}A = I$
$(A^{-1})AB = AC(A^{-1})$
$IB = IC$
$B = C$

Not all matrices have an inverse. If the inverse exists, you can divide & get $B = C$. If not, you're SOL.

X-FILES

MULDER
SCULLY

SKINNER
DOGGETT

A is invertible if there exists B s.t. $AB = BA = I_n$

$Ax = b$
$x = A \backslash b$
$inv(A) \times b = x$

A singular matrix means no inverse.
Non-singular means it has an inverse.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} b & -b \\ -c & a \end{bmatrix}$$

$ad - bc \neq 0$

$A^{-1} = B$

$(A^{-1})^{-1} = A$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^{-1})^T = (A^T)^{-1}$

$B \cdot A = (A^{-1})A = I$

so A is the inverse of B

Doing rref is like transformations. In the end, you end up with I.

If we can find the individual transformations, we can find the original matrix.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

only works if you multiply on the left

An $n \times n$ matrix A is invertible if only if A is row equivalent to I_n.

1st prove that if A is invertible, it is row equivalent to I_n.

rref(A) has a pivot in every row so $= I_n$

Then prove that if A is row equivalent to I_n, it can be inverted.

If A is row equivalent to I_n, there exist elementary matrices $E_1, E_2 \dots E_p$ such that $E_p \dots E_2 E_1 A = I_n$ (those intermittent matrices you get after transformations.

Elementary matrices are invertible, so $(E_p \dots E_2 E_1)^{-1}$ exists

left-multiply both sides of equation by it:

$$(E_p \dots E_2 E_1)^{-1}(E_p \dots E_2 E_1)A = (E_p \dots E_2 E_1)^{-1} I_n$$

$$I_n A = (E_p \dots E_2 E_1)^{-1} I_n$$

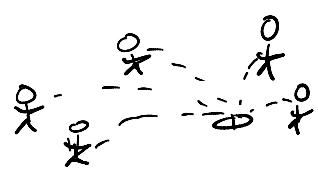
$$A = (E_p \dots E_n E_1)^{-1}$$

$[A \; I]$ perform row reduction

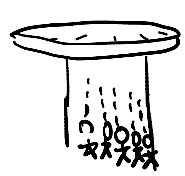
If A is row equivalent to I,

$[A \; I]$ is row equivalent to $[I \; A^{-1}]$

Otherwise, A does not have an inverse.



$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right]$$



Determinants

Thursday, November 02, 2006
8:03 AM

Determinants:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\left[a_{22}a_{33} - a_{23}a_{32}\right] - a_{12}\left[a_{21}a_{33} - a_{23}a_{31}\right] + a_{13}\left[a_{21}a_{32} - a_{22}a_{31}\right]$$

If $Ax = B$ Goes from $\mathbb{R}^n \to \mathbb{R}^m$
$A^{-1}b = x$ Goes from $\mathbb{R}^m \to \mathbb{R}^n$

If $Ax_1 = b$
$Ax_2 = b$ $x_1 \neq x_2$
A is not 1-1

$T^{-1}(b) = \nearrow x_1$, $\searrow x_2$

So getting 2 answers for this one thing doesn't make sense. b cannot be mapped to 2 things. Transformations cannot look like this.

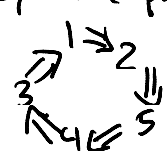
MULDER • 97 ILIUM

If a transformation is not onto, it doesn't work. Transformations only work for 1-1 or onto or both.

$1 \Rightarrow 2 \Rightarrow 5 \Rightarrow 4 \Rightarrow 3 \Rightarrow 1$

If there were 5 statements, and you can prove if 1 is true, 2 is true, etc. If you prove all of this you don't have to do anything else. You know $5 \Rightarrow 2$, or $4 \Rightarrow 5$, or $1 \Rightarrow 5$ etc.

Come up w/ a proof to put them in a ⊙.



Condition # is the stability of a matrix. Depending on round-off error, we can't depend on our solution. It could be wrong. Ill conditioned is a matrix w/ large round-off error, and a large condition #, and therefore it is close to being singular.

$x = Cx + d$ $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ We're assuming for one sector i, d_i is

$\begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ instead $d_i + \theta$.

$$X = (I - C) \backslash d$$

$$X = \begin{bmatrix} s_{11} & \cdots & s \\ s & & \\ \vdots & & \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} s_{11} & \cdots & s \\ \vdots & \ddots & \vdots \\ & & s_i \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_i + \theta \\ d_n \end{bmatrix}$$

$$= \begin{bmatrix} \quad \end{bmatrix} \left(\begin{bmatrix} d_1 \\ d_2 \\ d_i \\ d_n \end{bmatrix} + \begin{bmatrix} \theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_i \\ d_n \end{bmatrix} + \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \theta \\ 0 \end{bmatrix}$$

$$= \quad '' \quad + \begin{bmatrix} \theta & 0 & s_{1i}\theta & 0 & 0 \\ 0 & 0 & s_{2i}\theta & 0 & 0 \\ 0 & 0 & s_{3i}\theta & 0 & 0 \end{bmatrix}$$

$$= \quad '' \quad + \begin{bmatrix} s_{1i} \\ s_{2i} \\ s_{3i} \end{bmatrix} \theta$$

$$= (I - C)^{-1} d + \theta * [\text{ith col of } (I - C)^{-1}]$$

If θ was positive, we produce a little more (external demand slightly higher) & overall production is higher for all sectors.

If θ was neg, overall production less b/c demand ↓.

$$\underbrace{[\_ \ \_ \ \_ \ \_ \ \_ \ \_ \ \_]}_{M} \underbrace{P}$$

Transformation

$$\underbrace{[* \ * \ * \ * \ * \ * \ *]}_{C}$$

<u>basic encryption</u>

a b ← easy to break

↓ ↓

b e

$MP = C$

$C \to P$?

$(MP)M^{-1} = CM^{-1}$

$P = CM^{-1}$

so to do this we need a matrix that's invertible.

Use ASCII code to convert message to a sequence of #s.
Have to represent whitespace as well

Have to represent whitespace as well

Arrange #s into a matrix N,
Use linear transformation to transform N into another matrix.
N's inverse must exist or have integer entries

s = 'This is a test!'

double (s) converts to ASCII
char (n) converts to characters.

M is the key

$n \times n$ n^x ?

[ ]

If $n \times n = 5 \times 5$, & we have 13 char, $\underbrace{5 \times 3}_{\text{add a few spaces}}$ means we have some extra entries.

numb = reshape(double (s), r, c); r×c matrix
r → 3, c → 5

dft matrix, this one's 3×5. If you had a 3×3 original, it would work.

numb = [84 115 115 32 115
104 32 32 116 116
105 105 . — —]

numb − 32 =
= spaces.

So then you get 0–94, w/ 95 possible codes.

Come up w/ an M,

M * (numb − 32) = [bunch of big #s.] So you have to bring them down to the original range, 0–94.

So scale everything based on multiples of 95.
ncode = mod (M * (numb − 32) \ 95) So then add 32 to get back to ASCII range.

ncode = mod (M * (numb − 32)), 95) + 32

Encryption Key

Friday, November 03, 2006
8:04 AM

encryption key

$3\times 3 \quad M \cdot S \longrightarrow$ ciphertext (c)

$M^{-1} c \Rightarrow S$

(3x3) $\boxed{1\times 13}$ can't multiply, so must reshape $\boxed{}$.

we have 13 elmts, & 1 has to become a 3, so it has to be 3x5 to contain all integers

$3\times \underline{\text{cols}} >= 13$

1. Find an M — M must be invertible. Inverse only contains integers, M only contains integers.

2. Figure out how to reshape — numb=reshape(double(s), 3,5)
3. Pad the text w/ extra spaces — ncode=mod(m*(numb-32),95)+32
   scode=reshape(char(ncode),1,15)
4. Multiply M (encryption key) & scale accordingly

To get the value back, we do the same thing

scode = 'blah'.
ncode=reshape(double(s), 3, 5)
numb = mod(inv(m)*(ncode-32), 95) +32
sorig = reshape(char(numb), 1, 15)

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix} \quad 1(1-0) - \cancel{0(x-0)} + \cancel{0(x^2-x)}$$

1

This matrix can be inverted.

$$\begin{bmatrix} 1 & x & x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \quad 1(1-0) - x(0-0) + x(0-0)$$

$= 1$

so this can be inverted too.

$M = (M_1 M_2)$
$M_1^{-1} M_2^{-1}$
$(M_2 M_1)^{-1}$

You come up w/ 2 big prime #s, p & q, of which M is the product.

Extra spaces must be manufactured with 1's.

$e \; m \; d = p*q$ public

using m & d, the aliens create a message. Mulder has e, & e is a secret. d is a well known alien text key, & now only Mulder has e & d. and he can figure

out that they're coming.

→ to create spaces: educational doodling

[plaintext spaces] = plaintext You can regard spaces as integers of ASCII value 32

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$ad - bc$

$ad - bc = 0$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -c/a & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

row of all 0's, and so you can stop there. no inverse.

$\underline{A^T inv} = A inv$ We want to prove that A is invertible.

if this is invertible,

$A^T B = BA^T = I$ B is the inverse, A^{-}

$(A^T B)^T = (BA^T)^T = I^T$ we just took $T(A^T B)$

$A \cdot B^T = B^T \cdot A = I$

B^T is the matrix C for which $CA = I$, and therefore A^T is invertible.

$DA = I$

$\Downarrow$

$Ax = 0, \to x = 0$

$D(Ax) = (0)D$

$(DA)x = 0$

$Ix = 0$

$x = 0$ ✓

Vector Space, null Space

Tuesday, November 07, 2006
8:02 AM

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} x$$

$Ax = b$

$c_1 x_1 + c_2 x_2 + \dots + c_n x_n = b$

A vector space is a collection of objects that can be added or scaled. (Added or multiplied by a vector). Not really a vector in the traditional sense.

A nonempty set V of objects called <u>vectors</u>, on which are defined two operations.

addition $\vec{u} + \vec{v} \in V$ for all $\vec{u}, \vec{v} \in V$

Scalar multiplication $c\vec{u} = V$ for all $\vec{u} \in V$ & $c \in F$ (F is a field such as $\mathbb{R}$.)

Addition is commutative

Addition is associative

Addition has an identity elmt. zero vector $\vec{0} \in V$ s.t. $\vec{u} + \vec{0} = \vec{u}$

Addition has an inverse elmt. for each $\vec{u} \in V$, vector $\vec{u} \in V$ s.t. $\vec{u} + (-\vec{u}) = \vec{0}$

Scalar multiplication is distributive over vector addition.

$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

Scalar multiplication is distributive over field addition.

$(c + d)\vec{u} = c\vec{u} + d\vec{u}$

Scalar multiplication is associative.

Scalar multiplication has an identity elmt.

$V = \{(x, y) : x > 0, y > 0)$ w/ standard addition & scalar mult. in $\mathbb{R}^2\}$.

Claimed that this is <u>not</u> a vector space. b/c $\vec{x}$ & $\vec{y}$ have to be positive, so we can't have an inverse elmt. for addition (no vector would add to get $\vec{0}$)

Also if you mult. by a neg. scalar you get something not in the set.

A set of polynomials w/ exactly degree 'n' & addition/mult.

You end up w/ something not in the set.

Convert signal 

digitize the sample & then reproduce it.

Addition makes 2 signals. 
Multiplication strengthens signal. 

Infinite vector b/c it can expand forever.

Subspace

A subset of vector space that is itself a vector space.

A subset ~~of~~ of a vector space V w/ properties:

- Zero vector of V also in H
- H is closed under vector addition $\vec{u}+\vec{v} \in H$ for every $\vec{u},\vec{v} \in H$
- H is closed under scalar multiplication. $c\vec{u} \in H$ for every $\vec{u} \in H$ & scalar c

A set is closed if 2 elmts of the set, ~~&~~ perform operation on them, is also an element in it.

Vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$

$\mathbb{R}^3 \rightarrow (x, y, z)$

$\mathbb{R}^2 \rightarrow (x, y)$

$\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$ when seen like this

X-FILES

$V = \{(x, y, z)$

$H = \{(x, y, 0)$ H is a subset of V now.

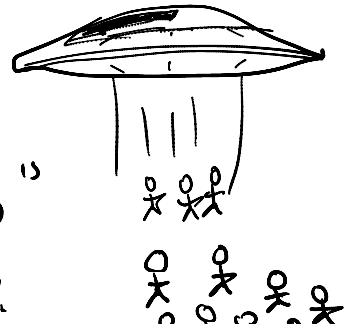
(0,0,0) belongs in H.

if you add any 2 elmts $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 0 \end{pmatrix}$

So closed under addition.

$c\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ 0 \end{pmatrix}$ And so this is still legit

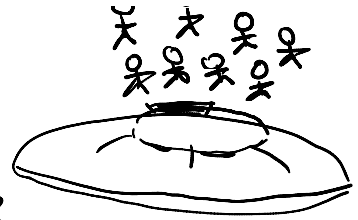
Therefore H is a subspace of V.



A line that does not pass through the origin is NOT a subspace. It must contain (0,0,0)

Every vector space is a subspace of itself.

Every vector space is a subspace of itself.



All polynomials of the form $p(t) = a \cdot t^2$, $a \in \mathbb{R}$
Is this a subspace of P_n?

$V = \{p(t) = at^2,\ a \in \mathbb{R}$

$p(d_1) + p(d_2)$

$$\begin{array}{r} x^3 + x^2 + 1 \\ (2x^3 - 2x^2 - 1) \\ \hline 3x^3 - x^2 \end{array}$$

Scalar mult: $2(x^3 + x^2 + 1) = 2x^3 + 2x^2 + 1$

$(\vec{p} + \vec{q})(d) = \vec{p}(d) + \vec{q}(d)$

$(c\vec{p})(t) = c\,\vec{p}(t)$

$V = \{$(All polynomials of degree at most n)

$H = \{p(t) = at^2,\ a \in \mathbb{R}$

Is H a subspace of V?

Closed under addition — 2 elmts in H added $\overset{\text{elmt in}}{=}$ H?

$at^2 + bt^2 = (a+b)t^2$

$b = -a \quad = 0t^2$ 0 belongs.

Closed under multiplication $= ca^{t^2}$

$= (ca)t^2$ so this works.

$H = \{$set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

Is this a subset of $M_{2\times 2}$?

It contains the 0 elmt for a, b, & d.

Add 2 matrices of this form will give you same kind of result. $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} a & b- \\ 0 & d- \end{bmatrix}$

set of elmts $\{v_1, v_2 \ldots v_k\} \in \mathbb{R}^n$ the set of all linear combinations of v_1, v_2, v_k, is a subspace of $\mathbb{R}^n$.

Furthermore: If $v_1, v_2 \ldots v_k$ are in a vector space V then $\mathrm{Span}(v_1, v_2, v_3 \ldots v_k)$ is a subspace of V.

Set of vectors written in matrix form $m \times n$ we can determine:

Null space

Column space

Row space

Nul A (set of all solns of equation $Ax=0$)

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$ Parametric vector form.

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\begin{bmatrix} x_1 - 7x_3 \\ 2x_4 - 4x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & 0 \\ 1 & 0 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The Null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$
- zero vector of $\mathbb{R}^n$ is also in Nul A $\quad Ax_1 = 0, Ax_2 = 0$
- Nul A is closed under vector addition $\quad A(x_1 + x_2) = Ax_1 + Ax_2 = 0$
- Nul A is closed under scalar mult.

The Column Space of a matrix A (Col A) is the set of all linear combinations of the columns of A.

Column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$
- zero vector of $\mathbb{R}^m$ in Col A
- Col A closed under vector addition
- Col A closed under scalar mult.

Col A = Span $\{a_1, a_2, \ldots, a_n\}$
- range of linear transformation $x \to Ax$ (set of all images)
- set of solns of the system.

Row space of a matrix A, Row A, is set of all linear combo's of rows of A.

Row space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$
- same reasons.

Vector math

Wednesday, November 08, 2006
8:00 AM

$V = \mathbb{R}^3$

$H \leq V$

$0 \in H$

H is closed under addition

H is closed under multiplication by scalar.

} These prove the statement.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$Ax = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Nul A

$Ax = 0$

Col A

Set of all vectors y that can be written in this form:

$$y = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + 2c_2 + 3c_3 \\ 4c_1 + 5c_2 + 6c_3 \end{bmatrix}$$

$y = \text{span}\{c_1\ c_2\ c_3\}$

$H = \{p(t) = a + t^2,\ a \in \mathbb{R}\}$

$(p+q)(t) = p(t) + q(t)$

$(cp)(t) = c \cdot p(t)$

Is this a subspace of P_n?

$0(t)$ always 0

prove these → $0(t) \in H$

$p(t), p_2(t) \in H \Rightarrow (p_1 + p_2)(t) \in H$

$p_1(t) \in H \rightarrow (cp_1)(t) \in H$

Given $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 2 & -1 \end{bmatrix}$

- is $[1; -1; 2]$ in Nul A
- is $[-4; 5]$ in Col A?

$\boxed{a + t^2}$ cannot be 0 for any t. The 0 polynomial

is a polynomial that is 0 no matter what t is. t could be 1 and it's not 0. at^2 would be a 0 polynomial

Nul A : solutions x of $Ax=0$
Col A : linear combinations of cols of A.

$$\begin{bmatrix} 1 & -3 & -2 \\ 4 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$ in Nul A? All vectors that are a solution to $Ax=0$

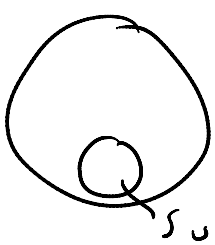
if $=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then yes. If something other than $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, no.

$$\begin{bmatrix} -4 \\ 5 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ -1 \end{bmatrix}$$ in Col A?

$$\underbrace{\begin{bmatrix} -4 \\ 5 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{x}\underbrace{\begin{bmatrix} 1 & -3 & -2 \\ 4 & 2 & -1 \end{bmatrix}}_{A}$$

$A \backslash b = x$ Simple to solve

IWANT TO BELIEVE



Subspace

Col A Span $\{a_1\ a_2 \ldots a_n\}$ columns of A

Every member of the subspace is a linear combination of these

$\mathbb{R}^2$ $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right.$ Any vector we pick in $\mathbb{R}^2$ can be expressed as $c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} = -3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ We want to come up w/ the smallest possible # of vectors that span the subspace. Every vector in the subspace is a linear combination of the vectors in the set.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right] \right\}$

Looking for a set of vectors linearly independent (so that it isn't a combination of the others), that spans H.

Nul A is set of all solns to $Ax=0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 3 \\ x_2 \\ x_3 \end{bmatrix}$$ free variables, x_2 & x_3

$$= \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

2 vectors that describe Nul A. B/c any soln x of $Ax=0$ can be written as a linear combination of these 2 vectors. These make up a basis for Nul A

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ 3 & 8 & 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With this, we know there are 2 non-pivot columns, so 2 vectors will be the basis for A.

$Ax=0$: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 - 5x_4 \\ 2.5x_3 - 1.5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6 \\ -2.5 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ -1.5 \\ 0 \\ 1 \end{bmatrix} x_4$$

basis for Nul A.

Col A =

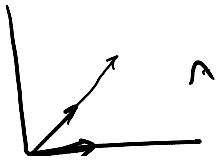
basis(col A) $\stackrel{?}{=} \{a_1, a_2 \ldots a_n\}$ if colums of a are not

basis(col(A)) $c_1, c_2 \ldots$

linearly independent this falls apart.

$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$ this is not a base.

$\mathbb{R}^2$



any vector in $\mathbb{R}^2$ can be written as a linear combination of these 3 columns, but we don't need that other arrow. Just the 2 are needed.

In the rref(A), each nonpivot column is a linear combination of the pivot columns.

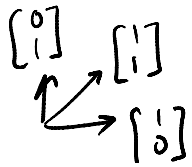
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ The 3rd column is a linear combination of the previous 2 $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x = 5\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$x = 7\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The vector set consisting of pivot col(A) is a basis (use the pivot columns of A, not of rref(A)).



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

H $\{b_1, b_2, \ldots b_n\}$

$x = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$ these coefficients are unique.

a diff set of coefficients $\{d_1, d_2 \ldots d_n\}$

$x = d_1 b_1 + d_2 b_2 + \ldots + d_n b_n$

$x = 0$

$$0 = \underbrace{(d_1 - c_1)}_{0} b_1 + \underbrace{(d_2 - c_2)}_{0} b_2 + \ldots + \underbrace{(d_n - c_n)}_{0} b_n$$

So $d = c_j$ • there is no other coefficient set.

Invertible matrix theorem

Thursday, November 09, 2006
8:03 AM

If A is $m \times n$
then nul A contains $n \times 1$ vector
while col A contains $m \times 1$ vector

nul A = solns of $Ax = 0$
col A = span $\{A$'s columns$\}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

basis
* lin. independent vectors
* Any vector can be written as a linear combination

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} c_2 + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} c_3$$

The number of elmts is called the <u>dimension</u> of the vector space.
All bases of a vector space contain the same # of elmts (dimension)

$$\begin{array}{c} \begin{array}{ccc} & & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & \boxed{0} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & \boxed{0} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} x_3$$

only one free variable. Dimension of nul A is 1.

For col A, the basis will be columns in original matrix that correspond to pivot columns. In this case, it's 2 of them.
Dimension col $A = 2$.

# of pivot columns = dimension of col A
# of free variables = dimension of nul A
dim col A + dim nul A = # of columns in the matrix

$$\dim(\text{col } A) = \text{rank } A$$

$$\text{rank}(A) + \dim(\text{nul } A) = n.$$

Invertible Matrix Theorem update
A is invertible *
Columns of A form a basis of $\mathbb{R}^n$
col $A = \mathbb{R}^n$
$\dim(\text{col } A) = n$
rank $A = n$
nul $A = \{0\}$
$\dim(\text{Nul } A) = \{0\}$

* $Ax=0$ has only the trivial solution. So rref(A) has a pivot in every column. This is why $\text{nul}A=0$

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 3 & 2 & 2 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -1 & 0 & -2 \\ 1 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$\dim(\text{col}A) = 3$
$\dim(\text{nul}A) = 1$
basis for $\text{col}A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

Choose original cols from original matrix, corresponding to pivot cols.

basis for $\text{nul}A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

$\dim(\text{col}A) = 3$
$\dim(\text{nul}A) = 0$
rank $A = 3$
basis for $\text{nul}A = \{\ \}$
basis for $\text{col}A =$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$$

Vectors span a plane
v_1 & v_2 are linearly independent
dimension of S is 2
how many vectors need to be added to $\{v_1, v_2\}$ to make a basis for $\mathbb{R}^3$ & what would they look like?

$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$ where $x_3 \neq 0$.

Every vector x in a vector space V has a unique representation as a linear combination of the elmts of basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$

$x = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$
$c_1, c_2, \dots, c_n$ are unique.

This allows us to regard the basis as a coordinate system w/ $(c_1, c_2, \dots c_n)$ the coordinates of vector x (relative to B)
we write $[x]_B = (c_1, c_2, \dots c_n)$ & call this the B-coordinate vector of x.

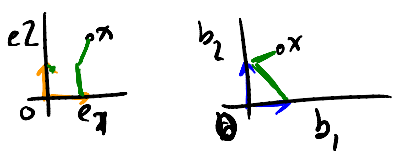
$$\begin{bmatrix} -2 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} -2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$c_1 = -2$
$c_2 = 7$

$-2, 7$ are the coordinates of the vector in the new system.

A vector $x = (3, 4)$, vector in $\mathbb{R}^2$.
Another basis of $\mathbb{R}^2 = B = \{(1,0), (1,2)\}$
B-coordinates of $x = (1,2)$ b/c $x = b_1 + 2b_2$



These should match up

Changing the basis hasn't changed anything. It has the same form as $\mathbb{R}^2$. That's called isomorphic to $\mathbb{R}^2$.

To find if x is in H, set up $A\{c_1, c_2\} = x$

$$v_1 c_1 + v_2 c_2 = x$$

$$\underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{3\times 2 \text{ matrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{x}_{3\times 1}$$

If you can find c_1, c_2 it would mean x is on the span $\{v_1, v_2\}$, & then you can find the B coordinates of x are c_1 & c_2 (b/c B is the basis of H).

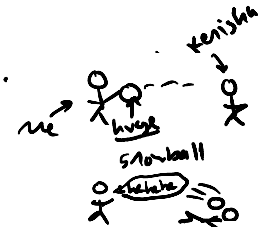
$$x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$x_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \cdot x_B = x$$

$$x_B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} x$$

Then you get back the original.



Geometrically, space $H = \text{Span}\{v_1, v_2\}$ <u>plane</u>.

$\mathcal{B} = \{1+t^2,\ t+t^2,\ 1+2t+t^2\}$ is a base for P_2.
Find coordinates vector of $p(t) = 1+4t+7t^2$ relative to $\mathcal{B}$.

$$1+4t+7t^2 = c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2)$$

$$1+4t+7t^2 = (c_1+c_3) + (c_2+2c_3)t + (c_1+c_2+c_3)t^2$$

$c_1+c_3=1 \qquad c_2+2c_3=4 \qquad c_1+c_2+c_3=7$

Solve & we're done.

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$x = v_1 + c_1 v_2 + c_2 v_3$$

STGOLLERS WOO!

Matrices

Thursday, November 09, 2006
8:03 AM

$$\begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ -2 & -2 & 2 & -8 & 8 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -11 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & 0 \\ -2 & -2 & 2 & 0 & 8 \\ 2 & 3 & 0 & 8 & 1 \\ 3 & 4 & -1 & 0 & -11 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 0 & 0 \\ -2 & -2 & 2 & -4 & 8 \\ 2 & 3 & 0 & 4 & 1 \\ 3 & 4 & -1 & -6 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 0 & 0 \\ -2 & -2 & 2 & -4 & 0 \\ 2 & 3 & 0 & 4 & 9 \\ 3 & 4 & -1 & -6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ -2 & -2 & 4 & -4 & 0 \\ 2 & 3 & -3 & 4 & 9 \\ 3 & 4 & -5 & -6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 \\ 2 & 3 & -3 & 1 & 9 \\ 3 & 4 & -5 & -11 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 4 & 0 & 0 \\ 2 & -3 & -3 & 1 & 9 \\ 3 & -5 & -5 & -11 & -1 \end{bmatrix}$$

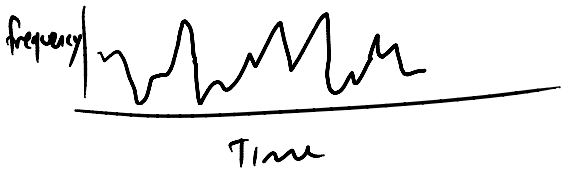
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -3 & 6 & 1 & 9 \\ 3 & -5 & 10 & 11 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 9 \\ 3 & -5 & 11 & -66 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ 3 & -5 & 11 & 66 & -100 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ 3 & -5 & 11 & 66 & -44 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ 3 & -5 & 11 & -44 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ 3 & -5 & 0 & -44 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & -5 & 0 & -44 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & -44 & 0 \end{bmatrix}$$

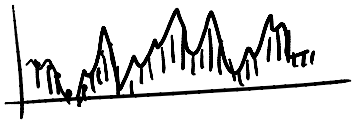
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 44 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & -44 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -44 & 0 \end{bmatrix}$$

Frequency, Time

Friday, November 10, 2006
8:02 AM



Sample each time & find frequency (partition).



So we can write it to a CD.

$t \to p(t)$
$t_2 \to p(t_2)$
$t_3 \to p_3$
$t_4 \to p_4$
$t_n \to p_n$

Multiply it by a constant c if you want it to sound louder.
$c > 1$ incr. volume
$c < 1$ dec. volume

2 signals, 2 pieces of music
$p_n \quad q_n$ if you add, you mix 2 signals together

$\{p_1\ p_2\ p_3 \ldots p_n\}$ signal 1

$\{q_1\ q_2\ q_3 \ldots q_n\}$ signal 2

$r = c_1 p + c_2 q$ new signal, part of the set of signals
$r = \{y_1, y_2, \ldots y_n\}$
basis of vector space $\{b_1 \ldots b_n\}$
p can be written like this
r can too
q can as well.

All signals can be written as linear combinations of the bases.

Sampling occurs 44,100 times per second. Times are very close together.

input A matrix
output: are ~~the~~ cols A linearly independent (T/F)
dim nul A
solution to $Ax=0$

```
function [isind, dimnul, sol] = indep(A)
%INDEP...
% help
    % k pivots
    % dim nulA = n-k
    % dim colA = k
    % if k=n
      % isind = true
      % else isind = false
```

% if A looks like this $\begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cc} x_2 & x_4 \end{array} \end{array}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 - x_2 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For each individual equation, we can express it in matlab

m(r, :).x = 0

$1x_1 + 1x_2 + 0x_3 + 1x_4 = 0$

Make all free variables 1. We don't know which ones are (assume we don't.)

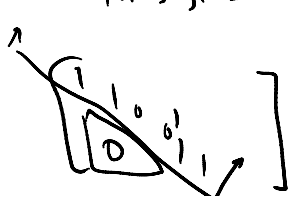
So $1(1) + 1(1) + 0 + 1(1) = 0$

$3 = 0$ $\emptyset$

By making $x_1 = 1$, we made a mistake. We're concentrating on finding x_1 so we have to revise it

$\boxed{-(m(r,:) \cdot x - 1)} = -1 + 1(1) + 1(1) + 0 + 1(1) = 0$

this gives us our x_1 value.



examine each row, but start at the diagonal.

function _____
% stuff
% stuff

% check arg.

% convert to rref

% num_pivots = 0

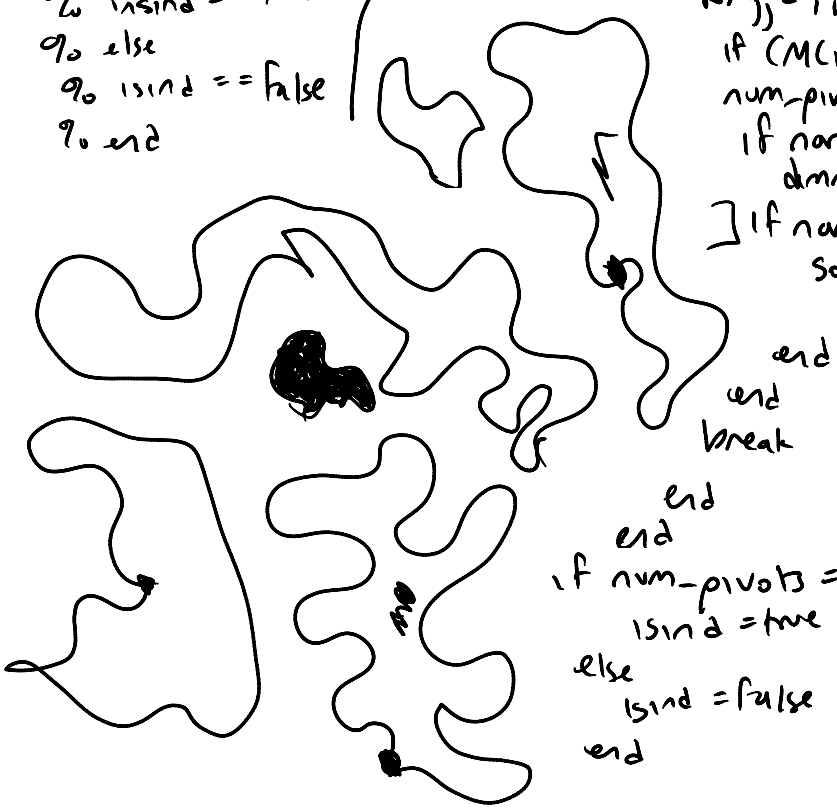
% dimnul = # cols A

% for each row
% start at main diagonal, going right
% check whether it's 1.
% if yes, update num_pivots & dimnul & solution.
% check num_pivots
% if == # of cols A
% isind == true
% else
% isind == false
% end

```
nargchk(1,1,nargin);
  error(msg);
nargchk(1,3,nargout)
  error(msg)
M = rref(A)
[rows cols] = size(A)
num_pivots = 0
  if (nargout > 1)
      dimnul = cols
        if (nargout > 2)
            sol = ones(cols, 1)
end
end
```

% only if user specifies that he wants it.

```
for ii = 1:rows
  for jj = ii:cols
    if (M(ii, jj) == 1)
    num_pivots = num_pivots + 1
     if nargout > 1
        dimnul = dimnul - 1
     if nargout > 2
        sol(jj) = 1 - M(ii, :) * ones(cols, 1)
end
end
break
end
end
if num_pivots == cols
    isind = true
else
    isind = false
end
```



Dot Product

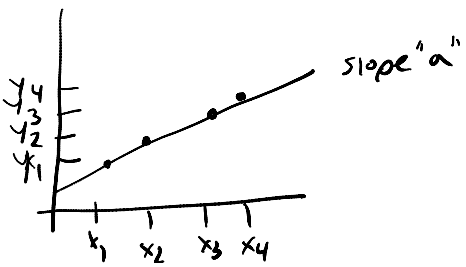
Tuesday, November 14, 2006
8:04 AM

$y = ax + b$
$y_2 = ax_2 + b$
$y_3 = ax_3 + b$
$y_4 = ax_4 + b$

$y - ax + b = 0$

If you go out & collect data, & get diff y's & x's.

4 equations & 2 unknowns



You're trying to find a. If the line isn't exactly on the points, then you have an error. It's the distance between the pt. and the line.

You want those distances, overall, to be as small as possible.

Inner Produt / dot product

$\vec{u} \cdot \vec{v} = \vec{u}_1\vec{v}_1 + \vec{u}_2\vec{v}_2 + \dots + \vec{u}_n\vec{v}_n$

$\vec{u} \cdot \vec{v} = \vec{u}^T\vec{v}$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$u^T = \begin{bmatrix} u_1 u_2 \cdots u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v \end{bmatrix} = \vec{v}$$

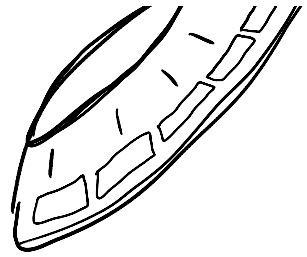
$\vec{u} \cdot \vec{u} \geq 0$

Length /norm of a vector $\vec{v}$ is defined as $\|v\| = \sqrt{\vec{v} \cdot \vec{v}}$
$= \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

A unit vector is a vector w/ length = 1
Normalizing a unit vector $\vec{v}$ means finding a unit vector $\vec{u}$ that has the same direction as $\vec{v}$

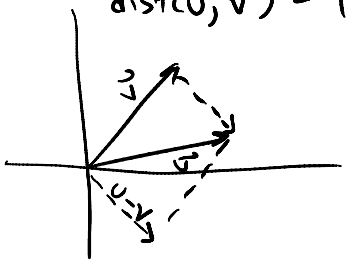
distance $= \sqrt{(v_1^2 + v_2^2 + v_3^2 \dots)^2} = 1,$

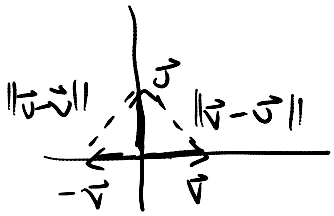
$v \quad \|v\|^2$



The distance between 2 vectors $\vec{u}, \vec{v}$ is the length of $\vec{u}-\vec{v}$

$$\text{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$





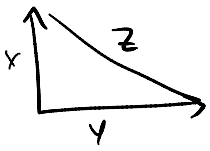
$\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|$
If $\perp$.

$$\|\vec{u}-v\| = \|\vec{u}+\vec{v}\|$$
$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$
$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$u\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot(\vec{u}+\vec{v}) + \vec{v}\cdot(\vec{u}+\vec{v}) \qquad \left((\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}\right)$$
$$\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$-\vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{v}$$
$$-2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v}$$
$$0 = 4\vec{u}\cdot\vec{v}$$
$$0 = \vec{u}\cdot\vec{v}$$

so they're $\perp$ when $\vec{u}\cdot\vec{v} = 0$

orthogonal

The zero vector is orthogonal to everything.



$$z^2 = x^2 + y^2$$
$$\text{distance } z = \sqrt{|x|^2 + |y|^2}$$

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 = \|y-x\|^2 = (y-x)(y-x)$$

0, $\perp$.

$$y\cdot y - 2y\cdot x + x\cdot x$$
$$\|y\|^2 + \|x\|^2 - 0$$

If a vector $\vec{u}$ is orthogonal to every vector in a subspace H of $\mathbb{R}^n$, then $\vec{u}$ is $\perp$ to H.

The set of all vectors $\vec{u}$ that are $\perp$ to H form the orthogonal complement of H, written $H^\perp$.

I LIKE
X-MEN
I DON'T

the orthogonal complement of H, written H!

$(\text{Row}A)' = \text{Nul}\,A \leftarrow$ solns to $Ax=0$

Rowspace (space spanned by rows of a matrix)

I DON'T LIKE MATLAB

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

All solutions x of $Ax=0$

So $(\text{Row}A)' = A\cdot x = 0$ b/c It's $\perp$.

$a_{11}\cdot x = 0$
$a_{21}\cdot x = 0$
$a_{31}\cdot x = 0$

$y = c_1 a_{11} + c_2 a_{21} + c_3 a_{31}$

$y\cdot x = (c_1 a_{11} + c_2 a_{21} + c_3 a_{31})\cdot x$

$= c_1(a_{11}\cdot x) + c_2(a_{21}\cdot x) + c_3(a_{31}\cdot x)$

$0 = c_1(0) + c_2(0) + c_3(0)$

Any vector you take in RowA is orthogonal to x b/c they're all 0, regardless of C's.

$(\text{Col}A^T)' = \text{Nul}A$

orthogonal complement

$\|\vec{u}\|\cos\theta$ projection of $\vec{u}$ onto $\vec{v}$

I WANT TO SEE X-FILES AGAIN!!!

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ when $\vec{u}$ & $\vec{v}$ are orthogonal, dot product $= 0$

when $\vec{u}$ & $\vec{v}$ are same vector, $\|\vec{u}\|$ & $\|\vec{v}\|$ are equal.

Let $\vec{v} = (a\ b)$ Describe vectors $\perp$ to $\vec{v}$.

$\begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix}$ $\quad ax + by = 0$

$a(x) = -b(y)$

x is a multiple of $-b$ $\quad$ y is a multiple of a

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}\cdot c$

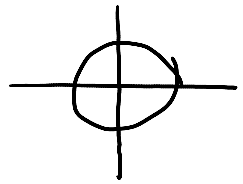
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} \cdot c$$

Orthogonal Projections

Wednesday, November 15, 2006
8:03 AM

$\{(x,y) : x^2+y^2=1\}$

$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$
$= \sqrt{x+x \cdot y+y} = 1$

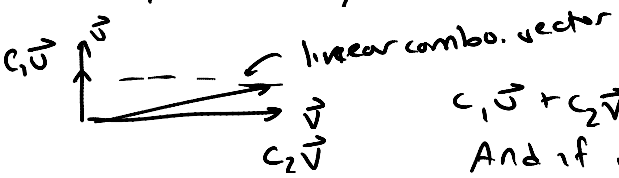


unit circle

0 is not in the set, b/c $0^2+0^2 \neq 1$
so it is not a subspace.

Let y be a vector orthogonal to u & v
Show that y is orthogonal to $\vec{u}+\vec{v}$

$\vec{y}\cdot\vec{u} = 0$
$\vec{y}\cdot\vec{v} = 0$
$\vec{y}(\vec{u}+\vec{v}) = \vec{y}\cdot\vec{u} + \vec{y}\cdot\vec{v} = 0+0=0$

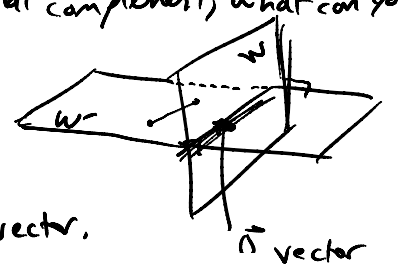


$c_1\vec{u} + c_2\vec{v}$ = linear combination vector
And if you do all linear comb.'s
then you get a plane.

If x is both in W & its orthogonal complement, what can you say about x?

$W \quad W'$
$\vec{u} \quad \vec{v}$

$\vec{u}\cdot\vec{v} = 0$
x must be $\vec{0}$ vector.



Let $\vec{u}=(5,-6,7)$ & W be the set of all vectors V in $\mathbb{R}^3$ s.t. $\vec{u}\cdot\vec{v}=0$. Is W a subspace of $\mathbb{R}^3$?
It's a Nul space. $Ax=0$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$
Given a set of vectors $V=(v_1, v_2, \dots, v_m)$ in $\mathbb{R}^n$ such that $v_1 \cdot v = 0$, for any 2 v_i $v_j \in V$
(Right hand rule, 3 directions 
Then V is called an orthogonal set.
Ex. $V=((0,0,1),(0,1,0),(1,0,0))$

$v = ((3,1,1), (1,2,1), (-\frac{1}{2}, 3, 5))$

Furthermore, V is linearly independent
This makes it a basis of $\mathbb{R}^n$
We call it an orthogonal basis.

We introduce the idea of basis b/c it's more efficient, & b/c we express any vector as a linear combination w/ unique coefficients, we have unique combo of coefficients

Let $\{v_1, v_2 \dots v_m\}$ be an orthogonal basis for a subspace H of $\mathbb{R}^n$. Then every elmt $x \in \mathbb{R}^n$ is written as linear combo

$$x = c_1 v_1 \dots c_m v_m$$

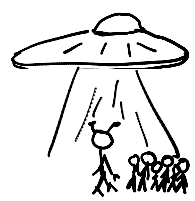
where $c_i = \dfrac{x \cdot v_i}{v_i \cdot v_i}$

The whole middle set will vanish if you multiply by v_i, except for $c_i v_i$

$$x \cdot v_i = c_1 v_1 \cdot v_i + c_2 v_2 \cdot v_i + \dots + c_m v_m \cdot v_i$$

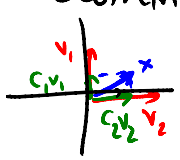
$$x \cdot v_i = c_i v_i \cdot v_i$$

$$\frac{x \cdot v_i}{v_i \cdot v_i} = c_i$$



$x = c_1 v_1 + c_2 v_2$ where $c_i = \dfrac{x \cdot v_i}{v_i \cdot v_i}$

Geometric representation

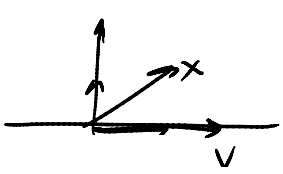


$c_1 v_1$ is the projection of x on v_2
This is a vector w/ same direction as v_1 & length $\|x\| \cos(90-\theta)$

From $x \cdot v_1 = \|x\| \|v_1\| \cos(90-\theta)$
we get that length of $c_1 v_1$ is $\dfrac{x \cdot v_1}{\|v_1\|}$

$$c \cdot v_1 = \frac{x \cdot v_1}{\|v_1\|} \cdot \frac{v_1}{\|v_1\|} = \frac{x \cdot v_1}{\|v_1\|^2} \cdot v_1 = \frac{x \cdot v_1}{v_1 \cdot v_1} \cdot v_1$$

Given a nonzero vector v, decompose a vector x into the sum of 2 vectors: one a multiple of v_1 & another orthogonal to v_1



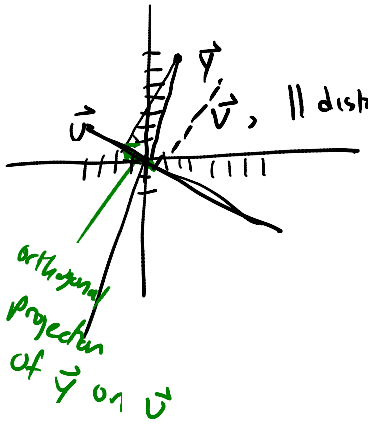
$x = \hat{v} + z$
where $\hat{v} = c\vec{v}$
& z is vector orthogonal to $\vec{v}$

where $v = c\vec{v}$
& z is vector orthogonal to $\vec{v}$

v & z are orthogonal, so will make up a basis. $x - \hat{v} = z$

Cuz they are streaming over the axes!

<u>Ex</u>. Let $\vec{y} = (1,7)$ & $\vec{v} = (-4,2)$. Compute orthogonal projection of y through $\vec{v}$ & the origin.



$\|$ distance between y & v $\|$

$y = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$y = c \cdot \vec{v} + \vec{v}$

$$\frac{y \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{y \cdot \vec{v}}{\|\vec{v}\|^2}$$

$y \cdot \vec{v} = -4 + 14 = 10$

$\|\vec{v}\|^2 = 16 + 4 = 20$

$c = \frac{10}{20} = \frac{1}{2}$

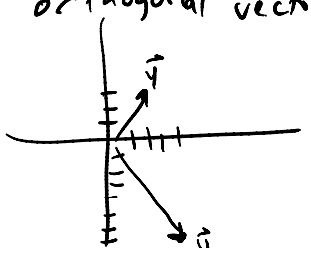
$(1,7) = \frac{1}{2}(-4,2) + \vec{v}$

$(1,7) = \underbrace{\boxed{(-2,1)}}_{\text{projection}} + \vec{v}$

Yeah!

Gee I wonder why

<u>Ex2</u> Let $y = (2,3)$ & $\vec{v} = (4,-7)$ write y as the sum of 2 orthogonal vectors, one in span(v) & one orthogonal to $\vec{v}$.



$\vec{y} = \vec{v} + w$

in span$\{v\}$

span of a single vector is a line, $c\vec{v}$

$V = \text{span}\{b_1, b_2 \ldots b_n\}$

$x \in V$

$x = c_1 b_1 + c_2 b_2 + \ldots c_n b_n$

$\vec{y} = (c\vec{v}) + \vec{w}$

$$\frac{y \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

then mult. this by $\vec{v}$

solve for $\vec{w}$.

Orthogonal Sets, Dot Products

Thursday, November 16, 2006
8:09 AM

Define orthogonal set $\{v_1, v_2 \dots v_n\}$ in $\mathbb{R}^n$ as a set of vectors whose pair-wise dot products are zero
(inner product)

If all vectors in an orthogonal set are also unit vectors, then it is called an orthonormal set. The vectors have been normalized.

$\{v_1, v_2\}$

$\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}\right\}$ still a basis, still linearly independent.

If an orthonormal set V spans a subspace H, then V is called the orthonormal basis of H

The dot product between every vector and itself is 1.

$$U^T = \begin{bmatrix} u_{11} & u_{23} \\ u_{12} & u_{22} \\ u_{13} & u_{23} \end{bmatrix} \quad \text{instead of } U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix}$$

$$U^T \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If a square matrix U has orthonormal columns, it is invertible & its inverse is just the transpose. We call that an orthogonal matrix

An $m \times n$ matrix U has orthonormal columns iff $U^T U = I$

If U is an $m \times n$ matrix w/ orthonormal columns, & $\vec{x}, \vec{y}$ are vectors in $\mathbb{R}^n$, then:

$$\|U\vec{x}\| = \|\vec{x}\|$$

transformation $x \to Ux$ preserves the length of x.

$$(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$$

$$(U\vec{x}) \cdot (U\vec{y}) = 0 \text{ iff } \vec{x} \cdot \vec{y} = 0$$

Preserves orthogonality of $\vec{x}$ & $\vec{y}$.

Orthogonal Vectors

Monday, November 20, 2006
8:01 AM

To see if 2 vectors are orthogonal, matrix multiplication

$$v = [v_1 \ v_2 \ v_3 \ \dots \ v_n]$$

$$v^T \cdot v = \text{dot product}$$

We have defined an orthogonal basis for subspace W as a basis for W that is also an orthogonal set.

$$c_i = \frac{x \cdot v_i}{v_i \cdot v_i}$$

If $\beta = \{v_1, v_2, \dots v_m\}$ is an orthogonal basis for subspace W of $\mathbb{R}^n$, those coefficients c are:

Every element x of W can be written as a unique linear combination of the elements of β

$$x = c_1 v_1 + c_2 v_2 \dots c_m v_m \quad \text{where } c_i = \frac{x \cdot v_i}{v_i \cdot v_i}$$

$z \cdot \hat{y} = 0 \leftarrow z \perp \hat{y}$

$z \in W^\perp$

y is decomposed into its components.

$\hat{y}$, orthogonal projection of y on the subspace is part of the set.

We can easily compute $\hat{y}$. All we need to know is an orthogonal basis of W. $y = \hat{y} + z$

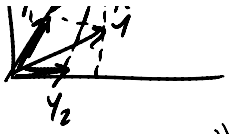
$$\hat{y} = \frac{v_1 (y \cdot v_1)}{v_1 \cdot v_1}$$

if $\beta = \{v_1\}$ = an orthogonal basis of W

W is a line in $\mathbb{R}^n$

If you're given 2 vectors:

$$\hat{y} = \hat{y}_1 + \hat{y}_2 = \frac{v_1 (y \cdot v_1)}{v_1 \cdot v_1} + \frac{v_2 (y \cdot v_2)}{\dots}$$



$v_2 \cdot v_2$

$\hat{y}_1$ is orthogonal projection of y onto span $\{v_1\}$

$\hat{y}_2$ is orthogonal projection of y onto span $\{v_2\}$

$\|z\|$ shortest distance between y & W and so $\hat{y}$ is best approx. $\hat{y}$ is y by elmts of W.

If $\beta = \{v_1, v_2\}$ is an orthonormal basis of W?

then $v_1 \cdot v_1$ & $v_2 \cdot v_2 = 1$

$\hat{y}_1 = v_1(y \cdot v_1)$

$\hat{y}_2 = v_2(y \cdot v_2)$

$\hat{y} = \hat{y}_1 + \hat{y}_2 = v_1(y \cdot v_1) + v_2(y \cdot v_2)$

$= [v_1 \ v_2][y \cdot v_1 \ \ y \cdot v_2]^T$

$[v_1 \ v_2][v_1 \ v_2]^T [y] = B \ B^T y$

$$\underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} - v_1 - \\ - v_2 - \\ - v_p - \end{bmatrix}}_{U^T} \begin{bmatrix} y \end{bmatrix}$$

$\text{proj}_W y = U \cdot U^T \cdot y$ where $\{v_1, v_2 \ldots v_p\}$ is an orthonormal basis of W

eh... i guess I'm listening?

When something is orthonormal, v_1 & v_2 are = to 1

$\vec{v}_1 = (1, 1, 0)$ $\vec{v}_2 = (-1, 1, 0)$, $y = (-1, 4, 3)$

Find orthogonal proj of y onto span $\{v_1, v_2\}$

Span $\{y, v_2\}$ is a plane

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ If v_1 & v_2 are dependent, it's a line. If they are

They are linearly independent, b/c one is not a multiple of the other. or

not a multiple of the other. or we could do $c_1 \vec{u}_1 + c_2 \vec{u}_2 = 0$ where the only soln for c_1, c_2 is the trivial one.

line. If they are linearly independent, it's a plane.

$\vec{u}_1 \cdot \vec{u}_2 = 0$ and so this is sufficient b/c if one was a multiple of the other you wouldn't get 0.

And they are also $\perp$, dot product being zero.

To find the projection, first normalize the vectors.

$\vec{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$\vec{u}_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$$\overbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}}^{U^T} \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\text{Proj}}$$

$v_1 = (1,1,0)$ $v_2 = (-1,1,0)$ $y = (-1,4,3)$ write y as the sum of a vector in $W = \text{Span}\{v_1, v_2\}$ & a vector in W.

$y = \hat{y} + z$ $\quad$ z is a vector in W

$y - \hat{y} = z$

$(-1,4,0) - (-1,4,3) = (0,0,-3)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} - \\ 4 \\ 0 \end{bmatrix}$$

$\|z\|$ = closest distance between y & W

$\text{sqrt}(0)^2 + \text{sqrt}(0^2) + \text{sqrt}((-3)^2) = 9$

$v_1 = (1,1,2)$ $v_2 = (-1,1,0)$ $y = (-1,4,3)$ Find orthogonal projection of y onto $W = \text{Span}\{v_1, v_2\}$

y is in W therefore its orthogonal projection will be y itself

z will be zero vector.

$v_1 = (1,1,-2)$ $v_2 = (5,-1,2)$, $y = (0,0,1)$

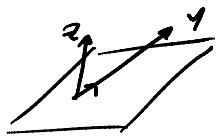
verify that y is not in the span of $W = \text{Span}\{v_1, v_2\}$

y is not a lin. comb. of v_1 & v_2: reduce augmented matrix of $[v_1 \ v_2 \ y]$

Use y to find a vector of v in $\mathbb{R}^3$ that is orthog. to u_2.

The zero vector is always orthogonal.
Besides the zero vector, the dot products of each of them is = to zero, & the dot products set equal to each other will find you a vector.

$v = c(0, 2, 1)$ where c is a scalar (free variable)



we can use y to find orthogonal vector on the space ($\hat{y}$) & then subtract to find z.

$y = \hat{y} + z$

So find $\hat{y}$ & then subtract from y.

Matlab

Given a matrix A, show columns are orthogonal
Normalize each column of A to get U

```
D = (A')*A =

%Normalize for ii = 1:size(A,2)
            U(:,ii) = A(:,ii)/norm(:,ii)
           end
```

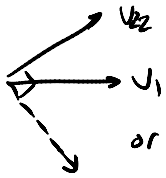
closest point to y in col U. $y = (1, 1, 1, 1, 1, 1, 1, 1)$
col U is space spanned by cols of U.
Cols of U are orthonomal set so you just have to multiply $U*U'*y$ = yhat in Matlab

Generating Orthogonal Basis

Tuesday, November 21, 2006
8:02 AM

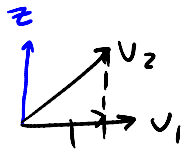
Generating an orthogonal basis: Given $\{u_1, u_2\}$ of a subspace W, how can we come up w/ an orthogonal basis for W?

Idea: We need to come up w/ a set $\{u_1$, vector orthogonal to $u_1\}$



or we could find a $\perp$ vector to u_2.

$\{v_1, v_2\}$ where $v_1 = u_1$, v_2 = vector orthogonal to u_1



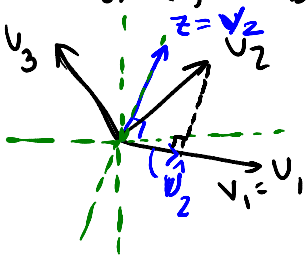
proj u_2 onto $u_1 = \hat{u}_2$

$u_2 = \hat{u}_2 + z$

$v_1 = u_1$

$$\boxed{v_2 = u_2 - \frac{u_2 \cdot v_1}{v \cdot v_1} v_1}$$

orthogonal basis $\{v_1, v_2, v_3\}$?



$\{u_1, u_2, u_3\}$

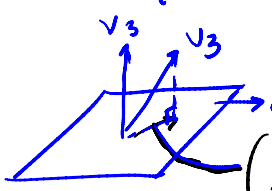
We're looking for, given $u_1, u_2, \& u_3$ the basis that contains $\{u_1$, vector orthogonal to u_1, vector orth. to $v_1 \& v_2\}$

b/c we want them to be an orthogonal set $\rightarrow$

Subspace span$\{u_1, u_2, u_3\}$

Space spanned by $\{v_1, v_2\}$

by $\{u_1, u_2\}$



$\rightarrow$ orth basis $\{v_1, v_2\}$

$$\left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2\right)$$

$u_3 = \hat{u}_3 + z \qquad z = v_3$

$$\left\{v_1 = u_1,\ v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1,\ v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2\right)\right\}$$

$v_1 = u_1$

$v_2 = u_2 - \text{proj}_{W_1} u_2$, $W_1 = \text{span}\{v_1\}$

$v_3 = u_3 - \text{proj}_{W_2} u_3$, $W_2 = \text{span}\{v_1, v_2\}$

span$\{u_1, u_2\}$

$v_2 = u_2 - \text{proj}_{W_1} u_2$, $W_1 = \text{span}\{u_1\}$

$v_3 = u_3 - \text{proj}_{W_2} u_3$, $W_2 = \text{span}\{v_1, v_2\}$
$\text{span}\{u_1, u_2\}$

$v_{k+1} = u_{k+1} - \text{proj}_{W_k} u_{k+1}$

$W_k = \text{span}\{v_1, v_2 \dots v_k\}$

This is the Gram-Schmidt Process for finding an orthogonal basis

A: lin ind. columns

$A = QR \rightarrow$ upper triangular invertible matrix

matrix whose columns form orthogonal basis of Col A

Want... excitement ... need... something... not... boring

$$\underbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & & v_n \end{matrix}}_{A} \qquad \underbrace{\begin{matrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mn} \\ \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & & v_n \end{matrix}}_{Q}$$

which we got like we did above.

$Q \rightarrow$ orthonormal basis of A.

$A = [v_1 \ v_2 \cdots v_n]$

vector in $\mathbb{R}^m$

$$Q = \left[\frac{v_1}{\|v_1\|} \ \frac{v_2}{\|v_2\|} \cdots \frac{v_n}{\|v_n\|} \right] \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [u_1]$$

$v_1 = u_1$

$v_2 = u_2 - \text{proj}_{W_1} u_2$

$A = QR$

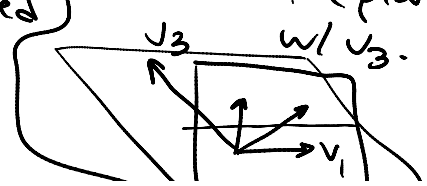
$u_1 = r_{11} v_1 + 0 v_2 + \dots 0 v_n$

$u_2 = r_{12} v_1 + r_{22} v_2$

v_2 is a lin. comb. of u_2 & $v_1 = u_1$

v_1 belongs in W_1 & in W_2 because it's just a line, intersecting w/ the plane created w/ v_3.

Last night, Mulder was:
almost abducted
kidnapped
brainwashed
returned



fooled into believing he was onto something rescued by scully.

$v_3 \in W_3$, just like $v_1 \in W_2$.

$v_3 \in W_3$ b/c $v_3 \in W_3$ & $\text{proj}_{W_2} v_3 \in W_3$ & $\in W_2$

$Q^T A = \underbrace{Q^T Q}_{I} R$

?? ?? ?? today's lecture

nodnod

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & 2 \\ 3 & -7 & 8 \end{bmatrix}$$

$v_1 = (3, 1, -1, 3)$

$v_2 = (-5, 1, 5, -7)$

$v_3 = (1, 1, -2, 8)$

One episode, Mulder was:
- Attacked by a genetic mutant
- Fell in a hole
- Lost his weapon
- Broke his arm
- Rescued by scully.

v_1, v_2, v_3

$v_1 = (3, 1, -1, 3)$

$v_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = v_2 - (-2) v_1 = (1, 3, 3, -1)$

okay... more than 1 episode.

$v_3 = v_3 - \left(\frac{v_3 v_1}{v_1 \cdot v_1} v_1 + \frac{v_3 v_2}{v_2 \cdot v_2} v_2 \right) = v_3 - \left(\frac{3}{2} v_1 + \frac{-1}{2} v_2 \right)$

$= (-3, 1, 1, 3)$

Another episode, Mulder was:
- Tricked by a psychic psycho.
- Shot
- Rescued by scully.

If you wanted to decompose A into Q & R, normalize it.

$v_1' = \left(\frac{3}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{3}{\sqrt{20}} \right)$

$v_2' = \left(\frac{1}{\sqrt{20}}, \frac{3}{\sqrt{20}}, \frac{3}{\sqrt{20}}, -\frac{1}{\sqrt{20}} \right)$

$v_3' = \left(\frac{-3}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{3}{\sqrt{20}} \right)$

$Q = [v_1' \ v_2' \ v_3']$

$R = Q^T A$

& $Q^T A = \begin{bmatrix} \frac{20}{\sqrt{20}} & -\frac{40}{\sqrt{20}} & \frac{30}{\sqrt{20}} \\ 0 & \frac{20}{\sqrt{20}} & -\frac{18}{\sqrt{20}} \\ 0 & 0 & \frac{20}{\sqrt{20}} \end{bmatrix}$

$v_1 \cdot v_1$

$v_1 \cdot v_1 = 20$

Least Squares

Tuesday, November 28, 2006
8:04 AM

Least Squares Problems

Overdetermined systems of equations.

Dimensions of A are $m \times n$ where $m > n$

A is a full rank matrix. Rank of A is n

Rank$(A) = n$

So it has a soln b/c there's a pivot in every column.

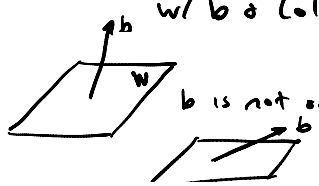
$Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

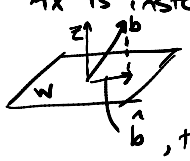
Col A span W

if b is not a lin. comb. of columns of A (no x that will make that true), then the relationship w/ b & Col A is



b is not on the plane.

Ax is instead $= b + r$, where r is error.

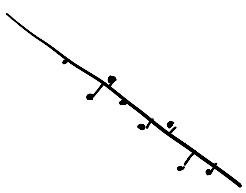


$\hat{b}$, the closest vector in W

$A\hat{x} = \hat{b}$

So we're looking for the solution to this system

z represents error that we have.



You have to minimize those distances.

$y_1 = ax_1 + b$

$y_2 = ax_2 + b$

$\vdots$

$y_8 = ax_8 + b$

Data fitting

Fit the given data to a function.

Polynomial interpolation.

$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$

to data $(x_1, y_1), (x_2, y_2) \ldots (x_m, y_m)$

1996 was 4th season of X-files.

$p(1996) = 1025$

$1025 = 1980 + c_1(1980) + c_2(1980)^2 + \ldots c_{n-1}(1980)^{n-1}$

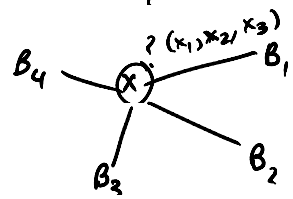
$\begin{bmatrix} 1 & 1980 & 1980^2 & \cdots & 1980^{n-1} \end{bmatrix} \begin{bmatrix} c_0 \end{bmatrix} \begin{bmatrix} 1025 \end{bmatrix}$

$$\begin{bmatrix} \cdots \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 999 \\ \vdots \end{bmatrix}$$

There will probably not be a soln b/c $m > n$, so find approx.

Navigation – given the distances between a pt. & known position, find location of the pt.

Numb3rs!
Charlie!



You have distance between B_1, B_2, B_3, & B_4, & you're trying to find x
3 equations, 3 unknowns, & you approximate.

The set of least squares soln to $Ax = b$ is the same as the non-empty set of solns of the <u>normal equations</u> $A^T A x = A^T b$. $A^T A$ is not necessarily invertible.

The columns of A are linearly independent the soln is unique & is given by $x_{LS} = (A^T A)^{-1} A^T b$

Prove
x_{LS} is a soln to the system $A^T A = A^T b$

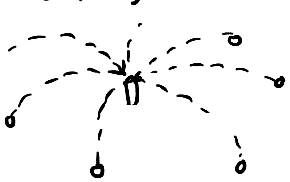
Given A & b, by the Best Approx Theorem, the orthogonal projection b_p of b onto Col A is the closest point in Col A to b.

$$\|b - b_p\| < \|b - y\| \text{ for any } y \neq b_p \text{ in Col } A.$$

Since b_p is in Col A, the sys $Ax = b_p$ has a soln x_{LS}
The above becomes $\|b - Ax_{LS}\| < \|b - Ax\|$. x_{LS} is a least squares soln.

By the Orthogonal Decomp. Theorem, the vector $b - Ax_{LS}$ is orthogonal to Col A, therefore $A^T(b - Ax_{LS}) = 0$

$$\Rightarrow A^T b - A^T A x_{LS} = 0$$
$$\Rightarrow A^T A x_{LS} = A^T b$$



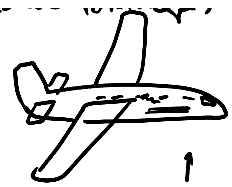
Prove
$A^T A$ is invertible
$A^T A x = 0$ has only one soln (which proves it.)

$$A^T A x = 0 \Rightarrow x^T A^T A x = 0$$
$$(Ax)^T A x = 0$$
$$\|Ax\|^2 = 0 \quad \text{(definition of length)}$$
$$Ax = 0 \quad \text{(if } \|x\| = 0\text{, then } v = 0)$$
$$x = 0 \quad \text{(b/c } A\text{'s columns are lin. indep.)}$$

Since A is invertible, the system ha

$x=0$ (b/c A's columns ...)

Since A is invertible, the system has a solution & it must be zero.



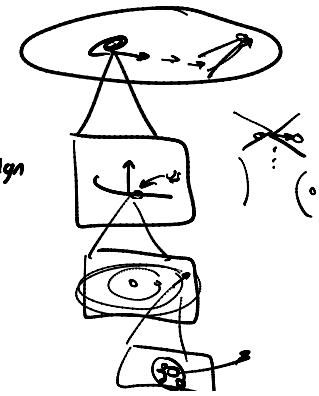
$\underbrace{y = x\beta + c}$

Linear model

x is called the design matrix

β is called the parameter vector

y is called the observation vector



one minute later

β_0 β_1 are regression coefficients when equation is a line & is called line of regression.

We want to avoid computing the inverse

If A is lin. ind. columns, apply Gram-Schmidt process w/ normalization to columns of A will make it possible to write A as product QR where Q is an $m \times n$ matrix whose columns form an orthogonal basis for col A. $Q^TQ=I$.

It is an upper triangular matrix which is invertible.

QR factorization.

$$A^TAx = A^Tb$$
$$(QR)^TQRx = (QR)^Tb$$
$$R^TQ^TQRx = R^TQ^Tb$$
$$R^TRx = R^TQ^Tb \quad (R \text{ is invertible})$$
$$Rx = Q^Tb$$

Since R is upper triangular, we can solve quickly & easily using back substitution.

Smallest error in set

Wednesday, November 29, 2006
8:08 AM

$y = Xb + \varepsilon$ where ε is error

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 \dots + b_{n-1} x^{n-1}$$

$$X = \begin{bmatrix} 1 & x & x_1^2 & \dots & x_1^{n-1} \\ \vdots & & & & \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} \leftarrow \text{unknown}$$

$y - Xb = \varepsilon$

$\|\varepsilon_1\|$
$\|\varepsilon_2\|$
$\|\varepsilon_3\|$

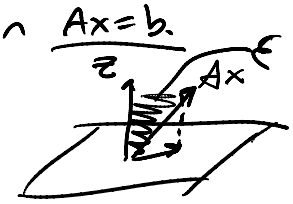
pick which one is smallest & choose that set of data b/c it has the smallest error

$y = \beta_0 f_0(x) + \beta_1 f_1(x) + \dots + \beta_n f_n(x)$
where $f_i(x)$ are linear or non-linear functions of x.

You have y & X & you need b.
$y = Xb$ however, X is not necessarily square. & so the system doesn't have a soln.
You need, instead, to find orthogonal projection of y onto Col X. Just like orthogonal projection of b onto Col A in $\underline{Ax = b}$.

$\hat{y} = Xb$
$\varepsilon = \hat{y} - y = \hat{y} - Xb$

$Ax = b$
$A\hat{x} = \hat{b}$ $\longrightarrow$ $\hat{b} - b = \hat{b} - Ax$



$z - Ax = \varepsilon$

You can do it this way or use normal equations $A^T A x = A^T b$
Solve for x.

$$x = (A^T A)^{-1} A^T b$$

You can do it this way or with QR factorization

$$A = QR$$
$$A^TAx = (QR)^T(QRx)$$
$$= R^T(Q^TQ)Rx$$
$$= R^TRx$$
$$A^Tb = R^TQ^Tb$$
$$Rx = Q^Tb$$

$A \to$ basis $\to$ orthogonal basis $\to$ normalize to get Q

Compute R

$Rx = Q^Tb$

$A = LU \to$ upper triangular matrix

$L \to$ unit lower triangular

$LUx = b$ $\quad Ux = y$

$Ly = b$

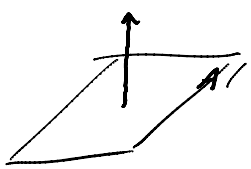
$\underline{\underline{A}} = \underline{\underline{Q}}R$ what is R?

Q has orthonormal columns

$$Q^TA = Q^TQR$$
$$\boxed{Q^TA = R}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$$

If $b = [1; 0; -1]$ what can you say about least squares soln of $Ax = b$? What is the relationship between b & columns of A?



The dot product between colums of A & b is 0, so b is orthogonal to both columns of A. The projection of b onto A is the 0 vector.

$Ax = \vec{0}$ will give you x_{LS}

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Consider $u = [1; -1/2]$ & $v[0; 1/2]$

Compute Au & Av

Could u or v be a least squares soln to $Ax = b$?

$$Au = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \to b_1 \quad \|b - b_1\|$$

Find the one w/

$$Av = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \longrightarrow b_2 \quad \|b - b_2\| \text{ the smallest error.}$$

$\|b - b_1\| = 6.16$

$\|b - b_2\| = 5.38$ so Av is closer to b than Au is. v might be a least squares soln.

If you compute Au & Av & the error is the same, then those solns are equidistant from b, & if there are 2 vectors equally close to b then they are not least squares solns $\Rightarrow$ those are unique.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Find the least squares soln for $Ax = b$

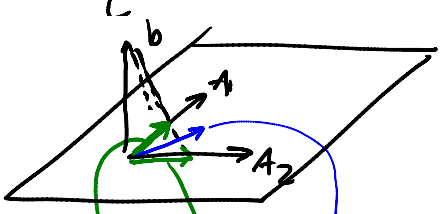
$A^T A = [3\ 0; 0\ 24]$ $\qquad A^T b = [9; 12]$

$3x_1 = 9 \Rightarrow x_1 = 3$

$24x_2 = 12 \Rightarrow x_2 = \frac{1}{2}$ $\qquad x_{LS} = [3; 1/2]$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Find projection of b onto Co A



this + this = $\text{proj}_A b$

$\text{proj}_{A_1} b + \text{proj}_{A_2} b = \text{proj}_A b$

$$\frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 = \text{proj}_A b$$

It is $b_p = A * x_{LS} = [\quad]$