Classifying Diff Eqns

Monday, March 26, 2007
8:59 AM

Comp lab Th. 10–1100

What's a diff eqn?

Eqn involving an unknown fn & its derivatives

e.g. $\frac{dx}{dt} + x = 0$ $\quad x = e^{-t}$, $x = 0$, $x = 3e^{-t}$
$x = Ce^{-t}$

$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$ $\quad m, b, k \geq 0$ damped spring
damping coeff. $= b$
spring constant $= k$

$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$ $\quad g =$ grav acc.
pendulum eqn $\quad \ell =$ length moment arm

<u>Terminology</u> for classifying diff eqns

<u>Order</u> – power/degree of highest derivative

<u>linear</u> or <u>nonlinear</u> – "power of unknown in eqn". If only 1st power appears of the unknown fn, then eqn is linear; otherwise nonlinear.

$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$ is nonlinear b/c of the $\sin\theta$

Systems of diff eqns

<u>ex</u> $\vec{F} = m\vec{a}$, 3D space $\quad \vec{a} = \frac{d^2}{dt^2}\langle x, y, z\rangle$
$\vec{F} = \langle f_1, f_2, f_3\rangle$

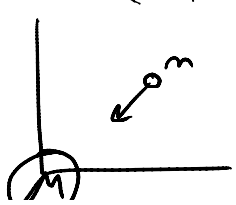
$f_1 = m\frac{d^2x}{dt^2}$

$f_2 = m\frac{d^2y}{dt^2}$

$f_3 = m\frac{d^2z}{dt^2}$

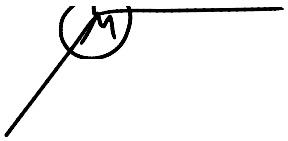
$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

$$= \left\langle -\frac{GMmx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{GMmy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{GMmz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right\rangle$$



$\vec{r} = \langle x, y, z\rangle$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$



$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$
$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$$

System of 3 second-order differential nonlinear eqns

Ordinary vs. Partial differential eqns (ODEs) (PDEs)

Has to do w/ 1 variable or 2 variables that fn depends on

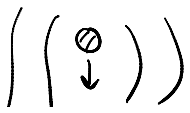
If unknown depends on >1 variable, PDE.

heat eqn

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \qquad T(x,t) = \text{temp}$$

First example

Spherical particle falling through highly viscous fluid



Forces - Gravity, $\underset{\text{viscous}}{\text{drag}}$, ~~buoyancy~~

Viscous drag depends on viscosity, temp, velocity

$$\text{Grav} = -mg\hat{j} \qquad F = ma = m\frac{d\vec{v}}{dt} = -m\frac{dv}{dt}\hat{j} = -mg\hat{j}$$

$$\vec{V} = -\underset{\text{speed}}{v}\hat{j}$$

For small enough particles $\vec{F}_{drag} \hat{=} \hat{j}\, 6\pi\mu a v$

Result due to Stokes

a = rad of sphere

μ = viscosity

v = speed

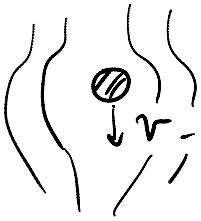
good approx for small Reynold's #

$$R = \frac{\rho a v}{\mu}$$

only true if $R \ll 1$

Euler's Approx

Tuesday, March 27, 2007
9:00 AM



v - speed of particle

ρ = density of fluid
a = radius of spherical particle
μ = viscosity of fluid

Assume: $\vec{F} = \vec{F}_{grav} + \vec{F}_{drag} \hat{=} m\vec{a}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-v\hat{j}) = (-\hat{j})\frac{dv}{dt}$$

$\hat{j}$] unit vector pointing in vertical direction.

$$\vec{F}_{grav} = mg(-\hat{j})$$

Assume Stokes' Approx to drag

$$F_{drag} \approx 6\pi\mu a v$$

Good approx if Reynold's # $= R = \frac{\rho a v}{\mu} \ll 1$

$$\vec{F}_{drag} = 6\pi\mu a v \hat{j}$$

$$-m\frac{dv}{dt}\hat{j} = -mg\hat{j} + 6\pi\mu a v\hat{j}$$

$$\boxed{\frac{dv}{dt} = g - \left(\frac{6\pi\mu a}{m}\right)v}$$

$\Leftarrow$ diff eq. for the speed of the particle v as a function of time

Solution depends on initial speed

"initial condition" at $t = t_0$ $\quad v(t_0) = v_0$

if v is pos it's going down
if v is neg it's going up

<u>Euler's Approx</u> Numerical soln

Based on the fact that $\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$

$$\frac{dv}{dt} \approx \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

If t is small

$$\frac{dv}{dt} = g - \underbrace{\left(\frac{6\pi\mu a}{m}\right)}_{\frac{1}{\tau}} v$$

$v(t_0) = v_0$ just b/c

t	v
t_0	v_0
$t_0+\Delta t$	$v_1 = v_0 + \left[g - \frac{v_0}{\tau}\right]\Delta t$
$t_0+2\Delta t$	v_2
$\vdots$	
$t_0+n\Delta t$	v_n
$t_0+(n+1)\Delta t$	v_{n+1}

$$v_1 = v(t_0+\Delta t) \approx \underbrace{v(t_0)}_{v_0} + \underbrace{\frac{dv}{dt}(t_0)}_{g-\frac{v}{\tau}}\Delta t$$

Taylor series but not higher order terms

Linear Approx

$$v_1 \approx v_0\left[g - \frac{v_0}{\tau}\right]\Delta t$$

Assume we have $v_n \approx v(t+n\Delta t)$

What is $v_{n+1} = v(t+(n+1)\Delta t) = v((t+n\Delta t)+\Delta t)$

Linearize v at time $t_n = t_0 + n\Delta t$

$$v_{n+1} \approx \underbrace{v(t_0+n\Delta t)}_{v_n} + \underbrace{\frac{dv}{dt}(t_0+n\Delta t)}_{g-\frac{v_n}{\tau}}\Delta t$$

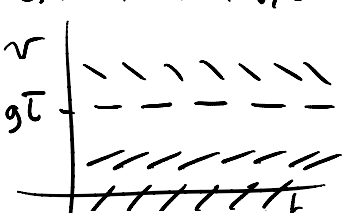
$$v_{n+1} = v_n + \left[g - \frac{v_n}{\tau}\right]\Delta t$$

$$v_2 = v_1 + \left[g - \frac{v_1}{\tau}\right]\Delta t$$

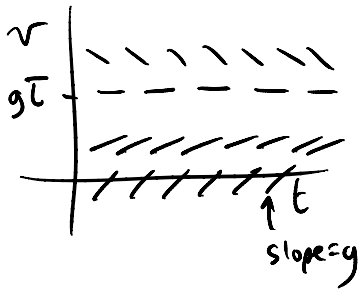
$\frac{dy}{dx} = f(y)$ or $\frac{dy}{dx} = f(y,x)$, $y(x_0) = y_0$

Qualitative Analysis of Solns

Based on direction field



$$\boxed{\frac{dv}{dt} = g - \frac{v}{\tau}}$$

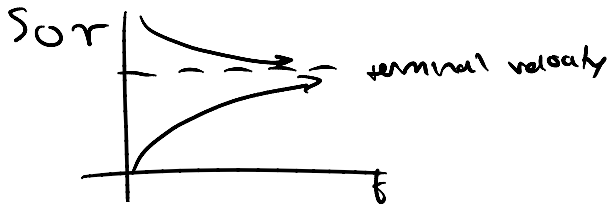


$$\boxed{\frac{dv}{dt} = g - \frac{v}{\tau}}$$

$v = 0 \Rightarrow \frac{dv}{dt} = g$

if $v = g\tau$, no slope

$\frac{dv}{dt} = g - \frac{v}{\tau}$ = slope of tangent line determined by direction field



Analytic Soln $v(t) =$ ___

$\frac{dv}{dt} = g - \frac{v}{\tau}$ ← separable 1st order differential eqn

$v(t_0) = v_0$

$\tau = \frac{m}{6\pi\mu a}$ = constant

Separable eqn soln method

$\frac{dy}{dx} = f(y)\, g(x)$ if $f(y)$ is 0, y isn't changing

$\int \frac{dy}{f(y)} = \int g(x)\, dx$ indefinite

$\frac{1}{g - \frac{v}{\tau}} \frac{dv}{dt} = 1$

$\frac{\tau}{g\tau - v} \frac{dv}{dt} = 1$

$\int \frac{1}{g\tau - v} \frac{dv}{dt} = \int \frac{1}{\tau}\, dt$

$\int \frac{1}{g\tau - v}\, dv = \int \frac{1}{\tau} dt$

$-\ln|g\tau - v| = \frac{t}{\tau} + C$

$$-\ln|gT-v| = \frac{t}{T} + C$$

$$\ln|gT-v| = -\frac{t}{T} - C$$

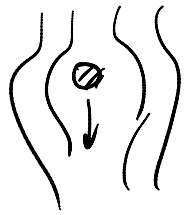
$$e^{\wedge} \qquad = e^{\wedge}$$

$$|gT-v| = e^{-\frac{t}{T}} \quad e^{-C} \text{ constant } K$$

$$gT - v = Ke^{-\frac{t}{T}}$$

Evaporating Snowball

Wednesday, March 28, 2007
9:01 AM



$$\frac{dv}{dt} = g - \frac{v}{\tau} \qquad \tau = \frac{m}{6\pi\mu a}$$

$v(0) = v_0$ initial speed downwards

$$\int g - \frac{1}{\tau}\frac{dv}{dt}\,dt = \int 1\,dt \quad \text{provided } v_0 \neq g\tau$$

Special soln $v(t) = g\tau$ (singular soln)

$$\left(-\frac{1}{\tau}\right)\int g - \frac{1}{\tau}\,dv = \left(-\frac{1}{\tau}\right)\int dt$$

$$\int \frac{1}{v - g\tau}\,dv = -\frac{t}{\tau} + C$$

$$\ln|v - g\tau| = C - \frac{t}{\tau}$$

$$v - g\tau = e^{C}e^{-\frac{t}{\tau}}$$

$$v = g\tau + Ke^{-\frac{t}{\tau}}$$

Check: $\frac{dv}{dt} = -\frac{1}{\tau}Ke^{-\frac{t}{\tau}}$

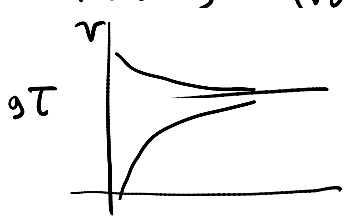
$$g - \frac{v}{\tau} = g - \left[g + \frac{K}{\tau}e^{-\frac{t}{\tau}}\right]$$

$$= -\frac{1}{\tau}Ke^{-\frac{t}{\tau}}$$

$t = 0$, $v = v_0 = g\tau + K$

$K = v_0 - g\tau$

$v(t) = g\tau + (v_0 - g\tau)e^{-\frac{t}{\tau}}$ when $t = 0$, $v(t) = v_0$



as $t \to \infty$, $v \to g\tau$
terminal

all solns decay exponentially toward terminal velocity.

$$v_{term} = g\tau = \frac{gm}{6\pi\mu a}$$

We assumed Stokes' eqn for viscous drag, which is valid if Reynold's # $R \ll 1$

$$R = \frac{\rho a v_{typical}}{\mu}$$

plug into this

$$= \rho a \frac{gm}{6\pi\mu^2 a} = \frac{\rho g m}{6\pi\mu^2} \ll 1$$

ρ → density of the fluid

m → density of the sphere $\left(\frac{4}{3}\pi a^3\right)$

$$\frac{\rho_{fluid}}{\mu^2}\left(\frac{4g\pi}{3\cdot 6\pi}\right)\left(\rho_{sphere}\, a^3\right) \ll 1$$

fluid properties

$$\frac{\rho_{fluid}}{\mu^2}\left(\frac{2}{9}g\right)\left(\rho_{sphere}\, a^3\right) \ll 1$$

$$a \ll \left(\frac{\mu^2}{\rho_f \rho_s}\right)^{\frac{1}{3}}\left(\frac{9}{2g}\right)^{\frac{1}{3}}$$

Evaporating snowball (spherical)

Assume evaporation rate is $\propto$ surface area S

Surface area S, Volume V $\quad \frac{dV}{dt}$ = evap rate

$\frac{dV}{dt} \propto S \qquad \frac{dV}{dt} = -KS \qquad K$ is a pos. constant

$$V = \frac{4}{3}\pi a^3 \qquad S = 4\pi a^2$$

$$a = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \qquad S = 4\pi\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$$

$$S = (4\pi)^{\frac{1}{3}}(3V)^{\frac{2}{3}}$$

initial condition, $V(0) = V_0$

$$-KS = -K(4\pi)^{\frac{1}{3}}(3V)^{\frac{2}{3}}$$

$$\frac{dV}{dt} = -\beta V^{\frac{2}{3}}$$

β is combo of $K(4\pi)^{\frac{1}{3}}(3^{\frac{2}{3}})$

$$V(0) = V_0$$

$$\int \frac{1}{V^{\frac{2}{3}}} \frac{dV}{dt} dt = \int -\beta dt \quad \text{provided } V_0 \neq 0$$

$$\int V^{-\frac{2}{3}} dV = \int -\beta dt$$

indefinite integral

$$3V^{\frac{1}{3}} = -\beta t + C$$

at $t=0$, $V=V_0$

$$3V_0^{\frac{1}{3}} = C$$

$$3V^{\frac{1}{3}} = -\beta t + 3V_0^{\frac{1}{3}}$$

$$V = \left(V_0^{\frac{1}{3}} - \frac{\beta}{3} t\right)^3$$

definite integral

$$\int_{V_0}^{V(t_f)} V^{-\frac{2}{3}} dV = -\beta \int_{t_0=0}^{t_f} dt$$

$$3V^{\frac{1}{3}} \Big]_{V_0}^{V_f} = -\beta t_f$$

$$3V_f^{\frac{1}{3}} - 3V_0^{\frac{1}{3}} = -\beta t_f$$

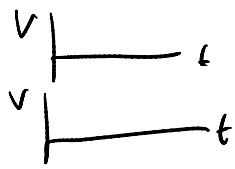
same thing

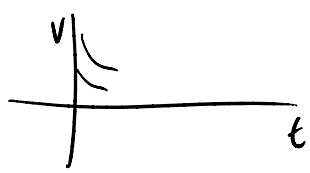
Volumes must be pos. So this soln valid for $t \leq \frac{3V_0^{\frac{1}{3}}}{\beta}$

$t = t_{nothing} = \frac{3V_0^{\frac{1}{3}}}{\beta}$ soln $= 0$ for $t > \frac{3V_0^{\frac{1}{3}}}{\beta}$

$V_0 = 1\,cm^3$

$V_0 = 10\,cm^3$





$$V(t) = \left(V_0^{\frac{1}{3}}\left(1 - \frac{\beta}{3}\frac{t}{V_0^{\frac{1}{3}}}\right)\right)^3$$

$$= V_0\left(1 - \frac{\beta}{3}\frac{t}{V_0^{\frac{1}{3}}}\right)^3$$

ratio of final to initial $\frac{V(t)}{V_0} = \left(1 - \frac{\beta}{3}\frac{t}{V_0^{\frac{1}{3}}}\right)^3$

all curves collapse to single curve

Existence & Uniqueness Theorem

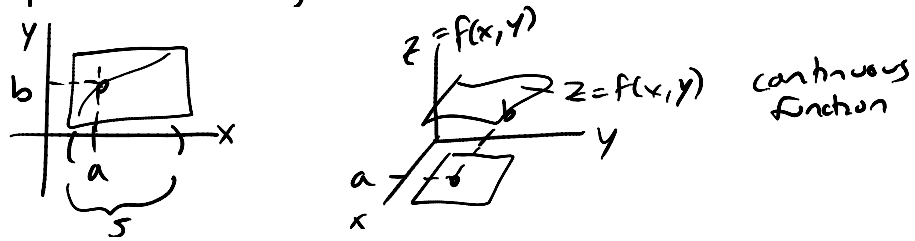
Friday, March 30, 2007
9:01 AM

p. 23 in book

Existence & Uniqueness Theorem for First Order ODEs (Ordinary Differential Eqns)

$$* \quad \frac{dy}{dx} = \underbrace{f(x,y)}_{\text{conditions on this fn so you get existence \& uniqueness}} \; ; \quad y(a) = b$$

<u>Thm</u> Suppose $f(x,y)$ is continuous on some rectangle in the x,y plane. containing (a,b) in its interior



continuous function

(1) Then (*) has at least 1 solution defined on some open interval J containing the pt $x = a$ [Statement says there exists a soln]

(2) If, in addition, the partial derivative $\frac{\partial f}{\partial y}$ is continuous on the rectangle then the soln is unique on some (perhaps smaller) open interval J_0 containing $x = a$ [Uniqueness]

Revisit Snowball Problem

$$\frac{dV}{dt} = -\beta V^{\frac{2}{3}} \quad , \quad V(t_0) = V_0$$

(1) $f(t,V) = -\beta V^{\frac{2}{3}}$ is a continuous function on a neighborhood (t_0, V_0)

exists a soln to initial value prob

(2) $\frac{\partial f}{\partial V} = -\frac{2}{3}\beta V^{-\frac{1}{3}}$ There's a prob at $V = 0$. Continuous everywhere except $V = 0$ so initial value problem (*) has a unique soln provided $V_0 \neq 0$

If $V_0 = 0$, there was no snowball. CSM would choose this soln. "<blows puff of smoke> There was no snowball"

<u>Section 1.5</u>

First order linear eqns & their solns

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)} \qquad y(a) = b$$

depending on p & q you can determine if soln exists & is unique.

$$\frac{dy}{dx} = q(x) - p(x)y = f(x,y)$$
$$-p(x) = \frac{\partial f}{\partial y}$$

<u>Ex. 1</u> Soln of falling bead problem using integrating factor

$$\frac{dv}{dt} = g - \frac{v}{\tau} \qquad \tau = \frac{m}{6\pi\mu a}$$

$$\frac{dV}{dT} = 1 - V$$

Nondimensionalizing

$$v = (g\tau)\,V(T)$$

units here

dimensionless

$$t = \tau\,T$$

unit here

$$\frac{dv}{dt} = \frac{d}{dt}(g\tau V) = g\tau\frac{dV}{dt}$$
$$= g\tau\frac{dV}{dT}\underbrace{\frac{dT}{dt}}_{\frac{1}{\tau}} = g\frac{dV}{dT}$$

$$g - \frac{v}{\tau} = g - \frac{g\tau V}{\tau}$$
$$g\frac{dV}{dT} = g(1-V)$$
$$\frac{dV}{dT} = (1-V)$$

$$\frac{dV}{dT} + V = 1$$

$$e^T\frac{dV}{dT} + e^T V = e^T \qquad e^T \text{ is integrating factor}$$

$$\int \frac{d}{dT}(e^T V)\,dT = \int e^T\,dT$$

$$e^T V = e^T + C$$
$$V = 1 + Ce^{-T}$$

<u>Ex. 2</u> $\frac{dy}{dx} + 2xy = 2x$ ← separable $\frac{dy}{dx} = 2x(1-y)$

but we'll solve w/ an integrating factor. "integrating factor" $\mu(x)$

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)2xy}_{\frac{d}{dx}(?)} = 2x\mu(x)$$

$$\frac{d}{dx}(?) = 2x\mu(x)$$

Try: $\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y$

This will work if $\frac{d\mu}{dx} = 2x\mu$

$$\mu = e^{x^2}$$

$$M\frac{dy}{dx} + 2xM\,y = 2xM \qquad M = e^{x^2}$$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2xe^{x^2}$$

$$\int \frac{d}{dx}\left(e^{x^2}y\right) = \int 2xe^{x^2}$$

$$e^{x^2}y = e^{x^2} + C$$

$$y = 1 + Ce^{-x^2}$$

$$\int \frac{dM}{dx}\frac{1}{M} = \int 2x\,dx$$

$$\ln|M| = x^2 + C$$

$$M = Ae^{x^2}$$

Soln of First Order ODE

Monday, April 02, 2007
8:56 AM

Soln of the general first order ODE using an integrating factor

$$(*) \quad \frac{dy}{dx} = p(x)y = g(x)$$

① Multiply eqn (*) by integrating factor $\mu(x)$, a function to be determined

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = g(x)\mu(x)$$

② Choose $\mu(x)$ so that the left-hand side is a derivative specifically

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y$$

$$L \quad = \mu\frac{dy}{dx} + \mu p y$$

Hence $\boxed{\frac{d\mu}{dx} = \mu(x)\,p(x)}$ ← eqn for the integrating factor μ

③ Solve for $\mu(x)$

$$\int \frac{d\mu}{\mu} = \int p(x)\,dx$$

$$\ln|\mu| = \int p(x)\,dx$$

$$\mu = e^{\int p(x)dx}$$

No arbitrary constant of integration b/c it drops out in the end.

④ Solve

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y = e^{\int p(x)dx}g(x)$$

Multiply by integrating factor

$$\Rightarrow \frac{d}{dx}\left[e^{\int p(x)dx}y\right] = e^{\int p(x)dx}g(x)$$

$$\int \frac{d}{dx}\left[e^{\int p(x)dx}y\right]dx = \int e^{\int p(x)dx}g(x)\,dx$$

$$e^{\int p(x)dx}y = \int e^{\int p(x)dx}g(x)\,dx + C$$

$$y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}g(x)\,dx\right] + Ce^{-\int p(x)dx}$$

Check this:

$$\frac{dy}{dx} = -p(x)y + g(x)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}g(x)dx\right] + Ce^{-\int p(x)dx}\right)$$

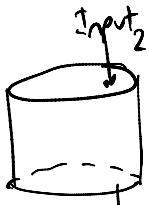
$$= e^{-\int p(x)dx}(-p(x))\left[\int e^{\int p(x)dx}g(x)dx\right] + \underbrace{e^{-\int p(x)dx}\left[e^{\int p(x)dx}g(x)\right]}$$

$$+ C\,e^{-\int p(x)dx}(\ldots)$$

$$+ Ce^{-\int p(x)dx}(-p(x)) \qquad \overbrace{\qquad}^{g(x)}$$

$$= -p(x)\underbrace{\left[e^{-\int p(x)dx}\int e^{\int p(x)dx} g(x)\,dx + Ce^{-\int p(x)dx}\right]}_{y \text{ (what we started with)}}$$

$$= -p(x)\,y + g(x)$$



Input 2 gallons/min of saltwater, 1 lb/gal of salt

let $s(t)$ = lbs/gallon salt in tank

$V(t)$ = gallons (salt)water at time t

Output salt water at rate = 3 gallons/min

Initially: 60 gallons pure H_2O in the tank

Find amt of saltwater in tank at time t

Find amt of salt in tank at time t

What is max amt salt in tank?

Concentration of salt in the liquid $= \frac{s}{V}$

We want diff eqns for $v(t)$ & $s(t)$ $\quad \frac{dV}{dt} = ? \quad \frac{ds}{dt} = ? \quad V(0) = 60$ gal, $s(0) = 0$ salt lbs

What is ΔV in time Δt?

$$\Delta V = (2\text{ gal/min})\Delta t - (3\text{ gal/min})\Delta t = (-1\text{ gal/min})\Delta t$$

$$t > 60 \Rightarrow V(t) = 0, \quad s(t) = 0$$

$$\frac{\Delta V}{\Delta t} = -1 \Rightarrow \boxed{\frac{dV}{dt} = -1, \quad V(0) = 60}$$

What is ΔS at Δt?

$$\Delta S = (1\text{ lb/gal})(2\text{ gal/min})\Delta t - \left(\frac{s}{V}\right)\left(\frac{3\text{ gal}}{\text{min}}\right)\Delta t$$

$$\frac{\Delta S}{\Delta t} = 2 - \frac{3s}{V} \Rightarrow \boxed{\frac{dS}{dt} = 2 - \frac{3S}{V}, \quad s(0) = 0}$$

$$\int \frac{dV}{dt}dt = \int -1\,dt$$

$$V(t) = -t + C$$

$$V(0) = 0 + C$$

$$60 = C$$

$$V(t) = 60 - t \quad 0 \le t \le 60$$

$$\frac{dS}{dt} = 2 - \frac{3S}{V} : 2 - \frac{3s}{60-t}$$ linear 1st order, but not separable

$$\frac{dS}{dt} + \frac{3S}{60-t} = 2 \qquad p(x) = \frac{3}{60-t} \quad g(x) = 2$$

$$\frac{dy}{dx} + p(x)y = g(x)$$

Solve by integrating factors

$$\mu = e^{\int \frac{3}{60-t}dt} = -3\ln|60-t| = e^{-3\ln|60-t|}$$

$$= \left(e^{\ln|60-t|}\right)^{-3}$$

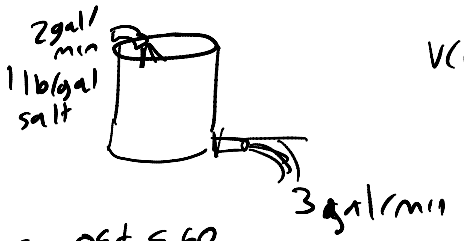
$$= (60-t)^{-3}$$

$$\mu = \left(\frac{1}{60-t}\right)^3$$

Salt Water Tank Prob, Population Prob

Tuesday, April 03, 2007
9:02 AM

Salt water tank Prob



$V(t)$ = salt water in gal in tank after t min

$s(t)$ = salt in lb in tank after t min

$V(0) = 60$

$s(0) = 0$

$\frac{dV}{dt} = -1$; $V(0) = 60$ $0 \le t \le 60$

$\frac{ds}{dt} = 2 - \frac{3s}{V}$; $s(0) = 0$

$V(t) = 60 - t$ $\qquad \frac{ds}{dt} = 2 - \frac{3}{60-t}s$, $s(0) = 0$

$\frac{ds}{dt} + \left(\frac{3}{60-t}\right)s = 2$ $\qquad$ Solve w/ an integrating factor

$$\mu = e^{\int \frac{3}{60-t}dt} = e^{-3\ln|60-t|} = \frac{1}{(60-t)^3}$$

$$\left(\frac{1}{(60-t)^3}\right)\frac{ds}{dt} + \frac{3}{(60-t)^4}s = \frac{2}{(60-t)^3}$$

$$\int \frac{d}{dt}\left(\frac{1}{(60-t)^3}s\right)dt = \int \frac{2}{(60-t)^3}\,dt$$

$$\frac{1}{(60-t)^3}s = \int \frac{2}{(60-t)^3}\,dt \qquad u = 60-t, \quad du = -dt$$

$$\frac{s}{(60-t)^3} = \frac{1}{(60-t)^2} + C$$

$t = 0,\ s = 0 \qquad 0 = \frac{1}{60^2} + C$

$\Rightarrow \qquad -\frac{1}{3600}$

$$s(t) = (60-t)^3\left[\frac{1}{(60-t)^2} - \frac{1}{60^2}\right]$$

$$\boxed{s(t) = (60-t) - \frac{(60-t)^3}{60^2}}$$

Max amt of salt, s_{max} occurs at time $t = t_{max}$

At $t = t_{max}$, $\frac{ds}{dt} = 0$

$$\frac{ds}{dt} = -1 + \frac{3(60-t)^2}{60^2}$$

$$0 = -1 + \frac{3(60-t)^2}{60^2}$$

$$\sqrt{1200} = \sqrt{(60-t)^2}$$

$$\frac{60}{\sqrt{3}} = 60 - t$$

$$60 - \frac{60}{\sqrt{3}} = t_{max}$$

$$25_{min} \approx t_{max}$$

$$S_{max} = s(t_{max}) = 60 - 60 + \frac{60}{\sqrt{3}} - \left(\frac{\left(\frac{60}{\sqrt{3}}\right)^3}{60^2}\right)$$

$$= \frac{60}{\sqrt{3}}\left(1 - \frac{1}{3}\right)$$

$$= \frac{40}{\sqrt{3}} \approx 23 \text{ lb}$$

Population growth models

Experimental Growth (too simplistic)

Let $P(t)$ = size of pop at time t. Continuous variable (treat as continuous real variable)

Let ΔP = change in pop size in amt time Δt

= # births in time Δt − # deaths in time Δt

Assume birth rate $\propto P \cdot \Delta t$ $\quad \Delta P_{birth} = aP\Delta t$

death rate $\propto P \cdot \Delta t$ $\quad \Delta P_{death} = -bP\Delta t$

$$\Delta P = \underbrace{(a-b)}_{k} P\Delta t \Rightarrow \frac{\Delta P}{\Delta t} = kP \to \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = kP$$

90% of the pop you meet in college

$$P(0) = P_0$$

$$P(t) = Ce^{kt}$$

$$P(0) = P_0 = C$$

$$\boxed{P(t) = P_0 e^{kt}}$$

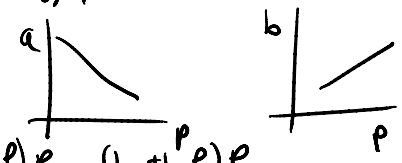
If k is pos, exponential divergence in population

Logistic Model Population growth

Assume birth and death rates (a & b) depend linearly on pop size

$$a \to a_0 - a_1 P \qquad a_0, a_1 > 0$$

$$b \to b_0 + b_1 P$$



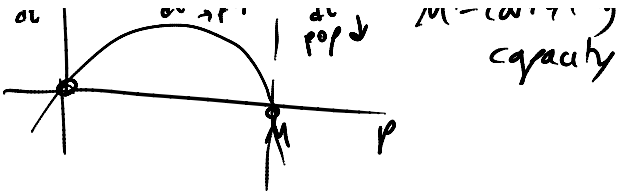
$$\frac{dP}{dt} = aP - bP = (a_0 - a_1 P)P - (b_0 + b_1 P)P$$

$$= (a_0 - b_0)P - (a_1 + b_1)P^2$$

$$= \underbrace{(a_1 + b_1)}_{K > 0} P \left[\underbrace{\frac{a_0 - b_0}{a_1 + b_1}}_{M > 0} - P\right]$$

$$\frac{dP}{dt} = KP[M - P]$$

$\frac{dP}{dt}$ | $\frac{dP}{dt} > 0$ pop ↑ | $\frac{dP}{dt} < 0$ | M = carrying


pop
capacity
M
P

Logistic Eqn for population

Wednesday, April 04, 2007
9:00 AM

Logistic eqn for pop

$$\frac{dP}{dt} = K(M-P)P$$

$K, M > 0$ (pos constants) $\quad P(0) = P_0$

M = Carrying capacity

Nondimensionalize $\quad \hat{P} = \frac{P}{M}$

$\tau = MKt$ dimensionless time

$$\frac{dP}{dt} = \frac{d}{dt}(M\hat{P}) = M\frac{d\hat{P}}{dt}$$

$$M\frac{d\hat{P}}{d\tau}\frac{d\tau}{dt} = M^2K\frac{d\hat{P}}{d\tau}$$

$$K(M-P)P = M^2K\frac{d\hat{P}}{d\tau} \qquad P = M\hat{P}$$

$$K(M-M\hat{P})M\hat{P} =$$

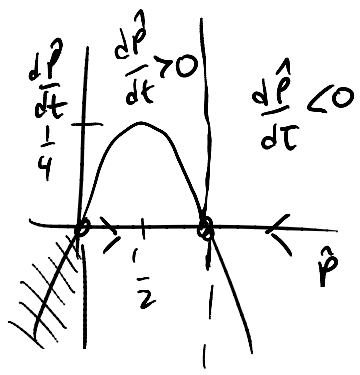
$$KM^2(1-\hat{P})\hat{P} = M^2K\frac{d\hat{P}}{d\tau} \qquad t = \frac{\tau}{MK}$$

$$\boxed{\hat{P} - \hat{P}^2 = \frac{d\hat{P}}{d\tau}}$$

$P(0) = P_0 = M\hat{P}_0$

$P(t) = P_0 e^{KMt}$

$$\frac{d\hat{P}}{d\tau} = (1-\hat{P})\hat{P}$$



Population can't be neg, so only pos. $\hat{P}$.

$\frac{d\hat{P}}{d\tau} > 0 \quad \frac{d\hat{P}}{d\tau} < 0$ regions

$\hat{P} = 0$ & 1 are equilib solns (singular)

expect $0 < \hat{P}_0 < 1 \Rightarrow$ soln increases toward $\hat{P} = 1$

$\hat{P}_0 > 1 \Rightarrow$ soln decreases towards $\hat{P} = 1$

Solve $\frac{d\hat{P}}{d\tau} = (1-\hat{P})\hat{P}$; $\hat{P}(0) = P_0 \neq 0, 1$

$$\int \frac{d\hat{P}}{(1-\hat{P})\hat{P}} = \int d\tau = \tau + C \qquad \int \frac{d\hat{P}}{(1-\hat{P})\hat{P}}$$

$$\frac{1}{(1-\hat{p})\hat{p}} = \frac{A}{(1-\hat{p})} + \frac{B}{\hat{p}}$$

$$\frac{1}{(1-\hat{p})\hat{p}} = \frac{A\hat{p} + B(1-\hat{p})}{(1-\hat{p})\hat{p}}$$

$$A\hat{p} + B(1-\hat{p}) = 1$$

$$\hat{p} = 0,\ B = 1$$

$$\hat{p} = 1,\ A = 1$$

OR

$$B + \hat{p}(A-B) = 1$$

$$\int \frac{1}{1-\hat{p}} + \frac{1}{\hat{p}}\, d\hat{p} =$$

$$-\ln|1-\hat{p}| + \ln|\hat{p}|$$

$$= \ln\left|\frac{\hat{p}}{1-\hat{p}}\right|$$

$$\ln\left|\frac{\hat{p}}{1-\hat{p}}\right| = \tau + C \Rightarrow e^{\ln\left|\frac{\hat{p}}{1-\hat{p}}\right|}$$

$$= e^{\tau + C}$$

$$= e^{\tau} e^{C}$$

$$\left|\frac{\hat{p}}{1-\hat{p}}\right| = \underbrace{e^{C}}_{K} e^{\tau}$$

$$\frac{\hat{p}}{1-\hat{p}} = K e^{\tau}\ ;\quad \tau = 0 \Rightarrow \hat{p} = \hat{p}_0 \Rightarrow \frac{\hat{p}_0}{1-\hat{p}_0} = K$$

$$\frac{\hat{p}}{1-\hat{p}} = \frac{\hat{p}_0}{1-\hat{p}_0} e^{\tau} \Rightarrow \hat{p} = \frac{\hat{p}_0}{1-\hat{p}_0} e^{\tau}(1-\hat{p})$$

$$\hat{p}\left[1 + \frac{\hat{p}_0 e^{\tau}}{1-\hat{p}_0}\right] = \frac{\hat{p}_0}{1-\hat{p}_0} e^{\tau}$$

$$\hat{p}(\tau) = \frac{\hat{p}_0}{(1-\hat{p}_0)\left[1 + \frac{\hat{p}_0 e^{\tau}}{1-\hat{p}_0}\right]} = \frac{\hat{p}_0 e^{\tau}}{1-\hat{p}_0 + \hat{p}_0 e^{\tau}} = \frac{\hat{p}_0}{(1-\hat{p}_0)e^{-\tau} + \hat{p}_0}$$

$$\hat{p}(t) = \frac{\hat{p}_0}{(1-\hat{p}_0)e^{-\tau} + \hat{p}_0}$$

Back to original dimensionless variable

$$p(t) = M\hat{p}(\underbrace{MKt}_{\tau})$$

$$p_0 = M\hat{p}_0$$

$$p(t) = \frac{M\hat{p}_0}{(1-\hat{p}_0)e^{-MKt} + \hat{p}_0} = \frac{p_0}{\left(1-\frac{p_0}{M}\right)e^{-MKt} + \frac{p_0}{M}}$$

$$p(t) = \frac{Mp_0}{(M-p_0)e^{-MKt} + p_0}$$

Check behavior of soln as $t \to \infty$

$$\lim_{t\to\infty} mp = \frac{Mp_0}{p_0} = M$$

$t \to \infty$ $\qquad$ P_0

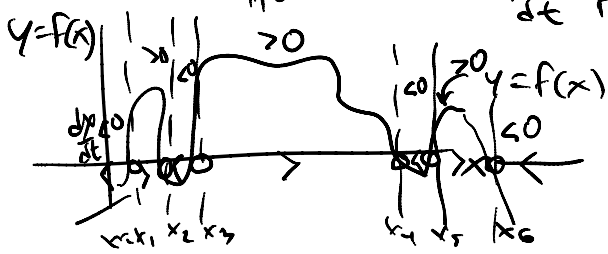
Qualitative Analysis of the following

$$\frac{dx}{dt} = f(x)$$

<u>Autonomous</u> 1st order DEQ

Autonomous means $\frac{dx}{dt}$ depends on x but not on t

Non Autonomous $\frac{dx}{dt} = f(x, t)$



places where $\frac{dx}{dt}$ changes sign

$x = x_j$ $\quad f(x_j) = 0 \Leftarrow$ equilib

$x_1, x_3, x_5,$ are unstable equilibria
x_2, x_4, x_6 are stable

$x = x_j$ is stable eq
if $f(x_j) = 0$ & $\frac{df}{dx}(x_j) < 0$

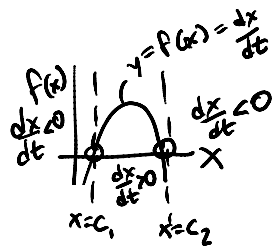
unstable if $\frac{dx}{dx}(x_j) > 0$

Linear stability of equilib solns

Autonomous First order Ordinary Diff eqns

$$\frac{dx}{dt} = f(x)$$

Qualitative Analysis

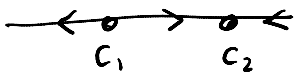


c_1 & c_2 are equilib solns, or critical pts

So when $f(x) = y = \frac{dx}{dt} = 0$, slope is 0.

$f(c_1) = f(c_2) = 0$

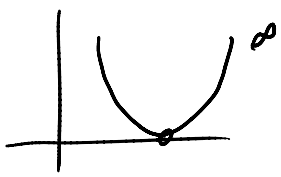
Phase diagram

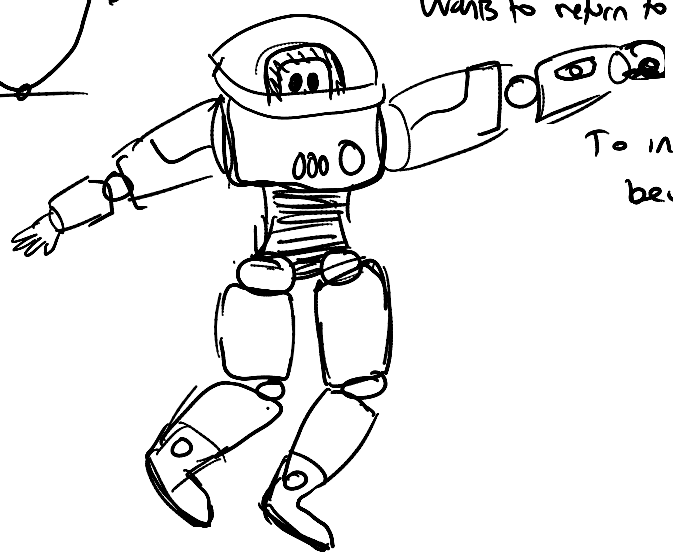


$x = c_1$ is an unstable equilib

$x = c_2$ is a stable equilib

Wants to return to starting pt





To infinity & beyond!

An equilib $x = c$ is <u>stable</u> provided that initial condition $x = x_0$ is sufficiently close to $x = c$, then $x(t)$ remains close to c for all $t > 0$

If condition is not met, it's unstable.

Sufficient conditions for stability/instability

Assume $x = c$ is an equilib soln. $f(c) = 0$

Now consider initial condition $x_0 = c + \tilde{x}$ $\quad 0 < |\tilde{x}| << 1$

$\frac{dx}{dt} = f(x) \qquad f(c) = 0 \qquad x_0 = c + \tilde{x}$

Approximate soln

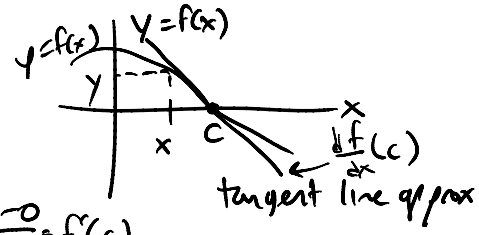
If $\tilde{x}(t)$ remains small then

$x(t) = c + \tilde{x}(t)$ $\qquad$ C is stable

LHS $\frac{dx}{dt} = \frac{d}{dt}(c+\tilde{x}) = \frac{d\tilde{x}}{dt}$

RHS $f(x) = f(c+\tilde{x})$

Approximate $f(x)$ by a straight line slope $\frac{df}{dx}(c)$



$y = f(x) \approx f'(c)(x-c)$ $\quad \frac{y-0}{x-c} = f'(c)$

$= f'(c)\tilde{x}$

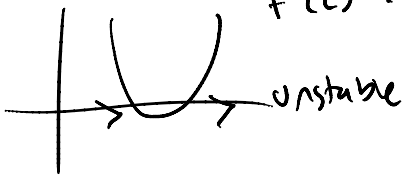
$\frac{d\tilde{x}}{dt} \approx f'(c)\tilde{x}$, $\tilde{x}_0 = \tilde{x}(0) \Rightarrow \boxed{\tilde{x} = e^{f'(c)t}}$

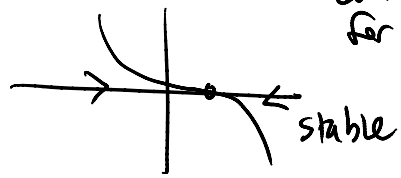
Sufficient condition for stability of $x=c$

$f'(c) < 0$

Sufficient condition for instability of $x=c$

$f'(c) > 0$





doesn't work for $f'(c)=0$

Taylor's Thm

Suppose the $(n+1)^{th}$ derivative of $f(x)$ exists on an interval containing a, b. Then $f(b) = f(a)$

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f^3(a)(b-a)^3}{3!} + \dots + \frac{f^n(a)}{n!}(b-a)^n + \frac{f^{n+1}(z)}{(n+1)!}(b-a)^{n+1}$$

z is between a & b

ex Expand $f(x) = \frac{1}{x}$ about $x=1$

$$f(x) = \frac{1}{x} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{3!}f^3(1)(x-1)^3 + \dots + \frac{1}{n!}f^n(1)(x-1)^n$$

$f(x) = \frac{1}{x}$ $\quad f'(x) = \frac{-1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$ $\quad f^3(x) = \frac{-6}{x^4}$

$f^4(x) = \frac{4!}{x^5}$ $\quad f^n(x) = (-1^n)\left(\frac{n!}{x^{n+1}}\right)$

$f(1) = 1$ $\qquad f(x) = \frac{1}{x} \approx 1 - (x-1) + (x-1)^2 - (x-1)^3 +$

$f(1) = 1$
$f'(1) = -1$
$f''(1) = 2$
$f^{3}(1) = -3!$
$f^{4}(1) = 4!$

$$f(x) = \frac{1}{x} \approx \underbrace{1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots + (-1)^n (x-1)^n}_{P_n(x)}$$

$$\left| \frac{1}{x} - P_n(x) \right| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| |x-1|^{n+1}$$

$$|(x-1)|^{n+1} \frac{\left| f^{n+1}(z) \right|}{(n+1)!} \leq \frac{M}{(n+1)!} |x-1|^{n-1}$$

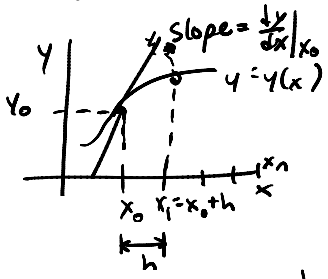
$$f^{(n+1)}(z) = (-1)^{n+1} \frac{n!}{z^{n+1}}$$

Solving 1st Order Diff Eqns

Monday, April 09, 2007
9:03 AM

Numerical Methods for solving 1st order diff eqns

$$* \begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$



Estimate $y = y(x)$ at discrete values of x

N

$$h = \frac{x_n - x_0}{n}$$

x	y
x_0	y_0
$x_0 + h = x_1$	$y_1 = y_0 + h f(x_0, y_0)$
$x_0 + 2h = x_2$	$y_2 = y_1 + h f(x_1, y_1)$
$\vdots$	
$x_0 + nh = x_n$	$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{\underbrace{x_1 - x_0}_{h}} = f(x_0, y_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Estimate how the error in Euler's method scales w/ h

Consider error in one step of integration

Let $y = \Phi(x)$ be the exact soln to *

$$y_{n+1} = \Phi(x_{n+1}) = \Phi(x_n + h)$$

$$\Phi(x_n) = \frac{dy}{dx}\Big|_{x_n} = f(x_n, y_n)$$

$$= \Phi(x_n) + \Phi'(x_n) h + \frac{1}{2}\Phi''(x_n) h^2 + \ldots$$

$$= y_n + h f(x_n, y_n) + E$$

Soln by Euler's Method + Error

Taylor's Thm allows us to estimate Error

$$E = \frac{1}{2} h^2 \Phi''(z)$$

$$\Phi(x_{n+1}) = \Phi(x_n + h) = \Phi(x_n) + h\Phi'(x_n) + \frac{1}{2} h^2 \Phi''(z)$$

z between x_n, x_{n+1}

$$|E_n| = C h^2 \quad \text{constant}$$

So the error is some constant of h^2

Total error $\propto N h^2$

integration steps

As $h \downarrow$, $N \uparrow$

$$N = \frac{x_n - x_0}{h}$$

$$TE \propto \frac{x_n - x_0}{h} h^2 \propto (x_n - x_0) h$$

$$E_n \propto h$$

error $\propto h$

if $2N$, then $\frac{1}{2}h$

EITHER/OR

IF/THEN

BEFORE/AFTER

Demonstrate this error scaling w/ prob. we can solve.

$$\frac{dy}{dx} = y$$

$$y(0) = 1$$

$$y(1) = ?$$

Exact soln

$\Rightarrow y(x) = Ce^x$

$1 = C$

$y(x) = e^x$

$y(1) = e = 2.718281\ldots$

$x_0 = 0 \quad y_0 = 1$

$x_1 = h \quad y_1 = y_0 + y_0 h = (1+h) y_0 = (1+h)$

$x_2 = 2h \quad y_2 = y_1 + y_1 h = (1+h) y_1 = (1+h)^2$

$x_3 = 3h \quad y_3 = y_2 + y_2 h = (1+h) y_2 = (1+h)^3$

$x_n = nh \quad y_n = y_{n-1} + y_{n-1} h = (1+h) y_{n-1} = (1+h)^n$

$$h = \frac{x_n - x_0}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$y(1) \simeq (1+h)^n, \; n = \frac{1}{h}$

h	n	approx	Error
.1	10	$y_1 \simeq 1.1^{10} =$ 2.59	2.59 − E = .124
.01	100	$y \simeq 1.01^{100} =$ 2.705	2.705 − E = .0135
.001	1000	2.717	.00136

Improve on Euler's Method

$$* \begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

1 subinterval from x_n to $x_n + h$

$$\int_{x_n}^{x_n+h} \frac{dy}{dx} dx = \int_{x_n}^{x_n+h} f(x,y)\, dx$$

$$y \Big]_{x_n}^{x_n+h} = y(x_{n+1}) - y(x_n) = y_{n+1} - y_n$$

$$= \int_{x_n}^{x_n+h} f(x,y)\, dx$$

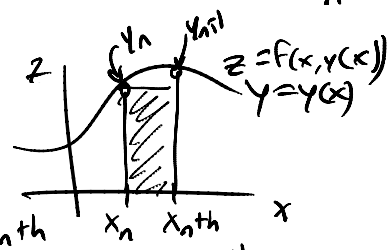
$$\boxed{y_{n+1} = y_n + \int_{x_n}^{x_n+h} f(x,y)\, dx}$$ exact

$$\boxed{y_{n+1} = y_n + \int_{x_n}^{x_n+h} f(x,y)\,dx}$$ exact

$$= \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

Left endpt approx

$$\int_{x_n}^{x_n+h} f(x,y)\,dx$$



$$\int_{x_n}^{x_n+h} f(x,y)\,dx \simeq \int_{x_n}^{x_n+h} f(x_n, y_n)\,dx$$

$$= f(x_n, y_n)\,h$$

Right end approx

Backward Euler Method

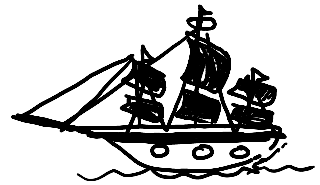
$$\boxed{y_{n+1} = y_n + h\,f(x_{n+1}, y_{n+1})}$$

implicit eqn for y_{n+1}

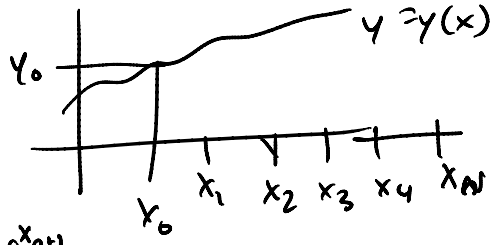
Improved Euler's Method

Tuesday, April 10, 2007
8:58 AM

First Order ODE's in particular
improvements to Euler's Method



$$\frac{dy}{dx} = f(x, y) \; ; \; y(x_0) = y_0$$



$$x_{n+1} - x_n = h$$

$$N = \frac{x_N - x_0}{h}$$

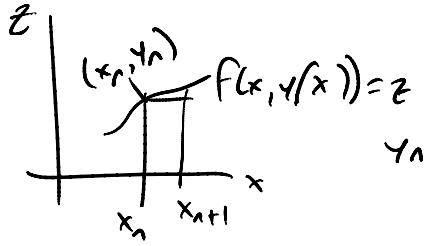
$$\int_{x_0}^{x_{n+1}} \frac{dy}{dx}\,dx = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

$$\underbrace{y(x_{n+1})}_{y_{n+1}} - \underbrace{y(x_n)}_{y_n} = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

Prob w/ this is we don't know $y(x)$

$$\boxed{y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)\,dx}$$ exact

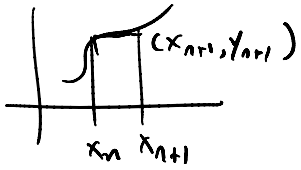
Euler's Method is a left-endpt. approximation to integral



$$y_{n+1} \approx y_n + f(x_n, y_n)\,h$$

Explicit formula for y_{n+1} in terms of x_n & y_n

Backward Euler Method : Right endpt. approx

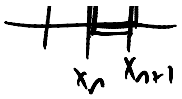


$$y_{n+1} \approx y_n + f(x_{n+1}, y_{n+1})\,h$$

Implicit formula for y_{n+1} in terms of x_n, y_n

Trapezoidal Approx to integral idea behind improved Euler method

$z = f(x, y(x))$



$$y_{n+1} \approx y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Improved Euler $\quad y_{n+1} \approx y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$

predictor $\rightarrow$ $u_{n+1} = y_n + hf(x_n, y_n)$

corrector $\rightarrow$ $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right]$

Improved Euler

Error scales as h^2, meaning that if you double N the error goes down by $\frac{1}{4}$

Example

$\frac{dy}{dx} = y$

$y(0) = 1$

$\Rightarrow y(x) = e^x$

$y(1) = e$

h	N	$(1+h)^N$ $y(1)$ Euler	error for Euler	$\left(1+h+\frac{h^2}{2}\right)^N$ $y(1)$ imp. Euler	error imp Euler
.5	2	2.25	.468	2.64	.078
$\frac{1}{4}$	4	2.441	.277	2.645	.023
$\frac{1}{8}$	8	2.566	.153	2.712	.0064

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$= y_n + hy_n = (1+h)y_n$$

$$y_{n+1} = y_n + \frac{h}{2}\left[y_n + (1+h)y_n\right]$$
$$= \left[1 + h + \frac{h^2}{2}\right]y_n$$

Euler is 1st order accurate for $E \sim h$

Improved Euler is 2nd order accurate $E \sim h^2$

4th order Runge Kutta $E \sim h^4$

$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1)$$
$$k_3 = f(x_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_2)$$
$$k_4 = f(x_n + h, y_n + hk_3)$$
$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

If $f(x,y)$ depends only on x, then this is just Simpson's approx. $\quad \frac{dy}{dx} = f(x)$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} \frac{dy}{dx}\,dx = \int_{x_n}^{x_{n+1}} f(x)\,dx = \frac{h}{6}\left[f(x_n) + 4f(x_n + \tfrac{h}{2}) + f(x_{n+1})\right]$$

Simpson's

RK4 $\quad y_{n+1} - y_n = \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$; $k_1 = f(x_n)$

$k_2 = f(x_n + \frac{h}{2})$

$k_3 = f(x_n + \frac{h}{2})$

$k_3 = f(x_n + \frac{h}{2})$
$k_4 = f(x_{n+1})$

Taylor Series Expansion

Wednesday, April 11, 2007
9:03 AM

4th order Runge Kutta $\quad (E \sim h^4)$

$$\frac{dy}{dx} = f(x,y) \qquad \begin{cases} k_1 = f(x_n, y_n) \\ k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{h}{2}k_1) \\ k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{h}{2}k_2) \\ k_4 = f(x_n + h, y_n + hk_3) \end{cases}$$

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

2 Special Cases

$$\frac{dy}{dx} = f(x) \Rightarrow y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x)\,dx$$

R-K is just Simpson's Approx of the integral in this case

$$\frac{dy}{dx} = f(y)$$

Show that $E \sim h^4$; We do this by showing that the error in 1 step (from y_n to y_{n+1}) is $E_n < C_n h^5$

Then the total error is $E_T < K \underset{\frac{(x_n - x_0)}{h}}{N} h^5 = K(x_n - x_0)\underline{\underline{h^4}}$

Idea behind proof that $E_n \sim h^5$

Approx soln: $y_{n+1} = y_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

"True soln": $y(x) = \Phi(x)$

$y_{n+1} = \Phi(x_n + h)$

Compare Taylor Series through h^4 terms

Approximate soln: Taylor Series Approx about (x_n, y_n)

$$k_1 = f(y_n)$$

$$k_2 = f(y_n + \tfrac{h}{2}k_1) = f(y_n) + f'(y_n)(\tfrac{h}{2}\overset{f(y_n)}{k_1}) + \tfrac{1}{2}f''(y_n)(\tfrac{h}{2}k_1)^2 + \tfrac{1}{3!}f'''(y_n)(\tfrac{h}{2}k_1)^3 + \tfrac{1}{4!}f^4(y_n)(\tfrac{h}{2}k_1)^4 + \dots$$

$$k_3 = f(y_n + \tfrac{h}{2}k_2) = f(y_n) + f'(y_n)(\tfrac{h}{2}k_2) + \tfrac{1}{2}f''(y_n)(\tfrac{h}{2}k_2)^2 + \tfrac{1}{3!}f'''(y_n)(\tfrac{h}{2}k_2)^3 + \tfrac{1}{4!}f^4(y_n)(\tfrac{h}{2}k_2)^4$$

$$k_4 = f(y_n + hk_3) = f(y_n) + f'(y_n)(k_3 h) + \tfrac{1}{2}f''(y_n)(k_3 h)^2 + \tfrac{1}{3!}f'''(y_n)(k_3 h)^3 + \tfrac{1}{4!}f^4(y_n)(k_3 h)^4 + \dots$$

Taylor Series expansion of approx soln $\qquad f = f(y_n)$

$$y_{n+1} = y_n + fh + \frac{1}{2} f' f h^2 + \frac{1}{6}(f'' f^2 + (f')^2 f) h^3 + \frac{1}{24}(4 f'' f' f^2 + (f')^3 f + f''' f^3) h^4 + \Theta(h^5)$$

"of order h^5"

$f, f', f'', \ldots$ are evaluated at y_n

Now Taylor expand the true soln $y = \Phi(x)$ $\left[\frac{dy}{dx} = \Phi'(x) = f(y) = f(\Phi(x))\right]$

$$y_{n+1} = \Phi(x_{n+1}) = \Phi(x_n + h) = \Phi(x_n) + \Phi'(x_n) h + \Phi''(x_n) h^2 + \frac{1}{3!}\Phi'''(x_n) h^3 + \frac{1}{4!}\Phi^{(4)}(x_n) h^4 + \ldots$$

$$= y_n + f(y_n) h +$$

$$\Phi''(x_n) = \frac{d}{dx}(\Phi'(x_n))\Big|_{x=x_n}$$

$$= \frac{d}{dx}(f(y(x)))\Big|_{x=x_n}$$

$$= f'(y(x)) \underbrace{y'(x)}_{f(y_n)}\Big|_{x=x_n} = f'(y_n) f(y_n) \checkmark$$

$$\Phi'''(x_n) = \frac{d}{dx}(\Phi'') = \frac{d}{dx}(f'(y_n) f(y_n)) = \frac{d}{dx}[f'(y_n)] f(y_n) + f'(y_n) \frac{d}{dx}(f(y_n))$$

$$= f''(y_n) \underbrace{\frac{dy}{dx}\Big|_{x=x_n}}_{f(y_n)} f(y_n) + f'(y_n) f'(y_n) \underbrace{\frac{dy}{dx}}_{f(y_n)}$$

$$\Phi \qquad = f'' f^2 + (f')^2 f \checkmark$$

$$\Phi^{(4)}(x_n) = \frac{d}{dx}(\Phi''')\Big|_{x=x_n} = \frac{d}{dx}[f'' f^2 + (f')^2 f]$$

2nd Order Diff Eqns

Friday, April 13, 2007
8:55 AM

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$
$$x_n = x_0 + nh \qquad h = x_{n+1} - x_n$$
$$y_n = y(x_n)$$

Euler $\quad y_{n+1} = y_n + h f(x_n, y_n)$

<u>Improved Euler:</u>

$$u_{n+1} = y_n + h f(x_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right)$$

<u>Runge Kutta</u>

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2}k_1\right)$$
$$k_3 = f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2}k_2\right)$$
$$k_4 = f(x_n + h,\ y_n + k_3)$$
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\frac{dy}{dx} = y \qquad y(0) = 1 \quad \Rightarrow \quad y = e^x$$
$$y_0 = 1$$
$$y_1 = y(x_0 + h) = y(h) = e^h \qquad \text{exact}$$
$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} = 1 + h + \tfrac{1}{2}h^2 + \tfrac{1}{3!}h^3$$

Euler $\quad y_1 = y_0 + h f(x_0, y_0)$

$f(x,y) = y$ for this example

$x_0 = 0$

$y_0 = 1$

$y_1 = 1 + h(y_0) = 1 + h$

Error in Euler is $O(h^2)$ "order h^2" means leading term is Ch^2

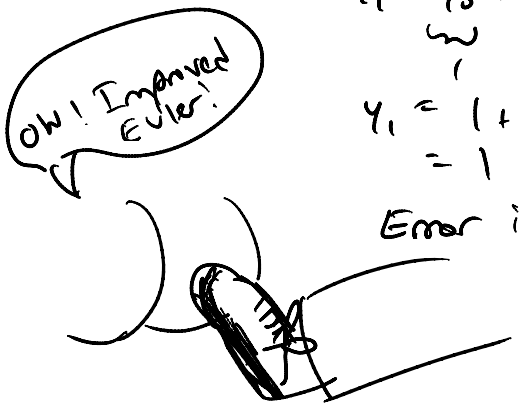
Improved Euler

$$U_1 = y_0 + h\underbrace{f(x_0, y_0)}_{y_0} = (1+h)\underbrace{y_0}_{1} = 1+h$$

$$y_1 = \underbrace{y_0}_{1} + \frac{h}{2}\Big(\underbrace{f(x_0, y_0)}_{y_0 = 1} + \underbrace{f(x_{n+1}, u_{n+1})}_{u_1 = 1+h}\Big)$$

$$y_1 = 1 + \frac{h}{2}(1+h)$$
$$= 1 + h + \frac{h^2}{2}$$

Error is $\Theta(h^3)$



Runge Kutta

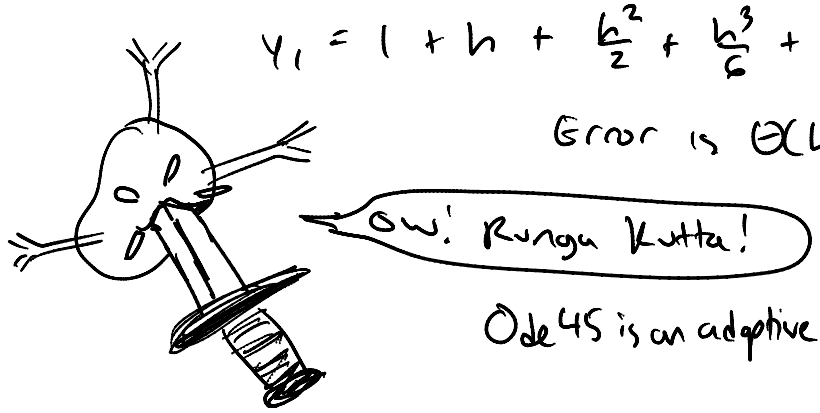
$$k_1 = f(x_0, y_0) = y_0 = 1$$
$$k_2 = f(x_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2}k_1) = y_0 + \tfrac{h}{2}k_1 = 1 + \tfrac{h}{2}$$
$$k_3 = f(x_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2}k_2) = y_1 + \tfrac{h}{2}k_2 = 1 + \tfrac{h}{2}(1 + \tfrac{h}{2})$$
$$= 1 + \tfrac{h}{2} + \tfrac{h^2}{4}$$
$$k_4 = f(x_0 + h, y_0 + k_3) = y_0 + hk_3$$
$$= 1 + h(1 + \tfrac{h}{2} + \tfrac{h^2}{4})$$
$$= 1 + h + \tfrac{h^2}{2} + \tfrac{h^3}{4}$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = 1 + \frac{h}{6}\left(1 + 2 + h + 2 + h + \frac{h^2}{2} + 1 + h + \frac{h^2}{2} + \frac{h^3}{4}\right)$$

$$y_1 = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{4!}$$

Error is $\Theta(h^5)$



Ode45 is an adaptive Runga Kutta

Chapter 3 Second order diff eqns

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

These eqns show up all the time in mechanics

These eqns show up all the time in mechanics

$$F = ma$$
$$F = m\frac{d^2x}{dt^2}$$

$$F(x, t, v) = F\left(x, t, \frac{dx}{dt}\right)$$

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right)$$

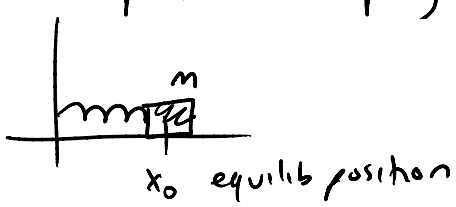
A necessary condition for a unique soln is to have 2 initial conditions eg $y(x_0) = y_0$ $\frac{dy}{dx}(x_0) = y_0$

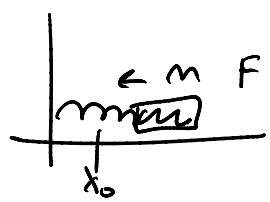
First examples

$$m\frac{d^2x}{dt^2} = F(x)$$

These conserve energy so energy conservation can be used as an initial condition

example mass on a spring obeying Hooke's law





$F = -kx$

k = spring constant, > 0

$$\frac{dx}{dt}\left[ma = m\frac{d^2x}{dt^2} = -kx\right]$$

$$m\frac{d^2x}{dt^2}\frac{dx}{dt} = -kx\frac{dx}{dt}$$

$$m\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right) = -k\frac{d}{dt}\left(\frac{x^2}{2}\right) = -k\left(\frac{2x}{2}\right)\frac{dx}{dt}$$

$$\frac{d}{dt}\underbrace{\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2\right]}_{\text{energy}} = 0$$

Conservation of energy eqn

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = E_0$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left(E_0 - \frac{k}{2}x^2\right)$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E_0 - \frac{k}{2}x^2\right)}$$

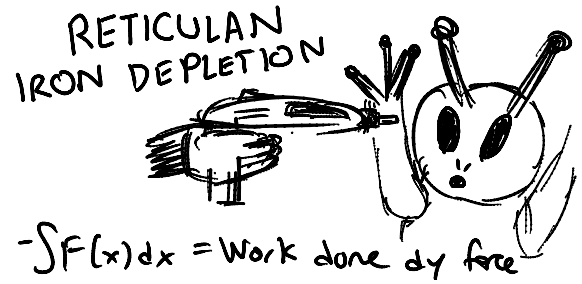
$$\int \frac{dx}{\sqrt{2E_0 - kx^2}} = \pm t + C$$

$$\int \sqrt{\frac{2E_0}{m} - \frac{k}{m}x^2} = \pm t + C$$

Potential E, Mass on a Spring

Monday, April 16, 2007
8:58 AM

$$m\frac{d^2x}{dt^2} = F(x)$$



express $F(x) = -\frac{dV}{dx}$

$V(x)$ = potential energy $= -\int F(x)\,dx$ = Work done by force

$$\underbrace{\int_{t_0}^{t_f} m\frac{d^2x}{dt^2}\frac{dx}{dt}\,dt}_{\int_{t_0}^{t_f}\frac{m}{2}\frac{d}{dt}\left(\left(\frac{dx}{dt}\right)^2\right)dt} = \int_{t_0}^{t_f} -\frac{dV}{dx}\underbrace{\frac{dx}{dt}\,dt}_{dx}$$

$$\int_{t_0}^{t_f}\frac{m}{2}\frac{d}{dt}\left(\left(\frac{dx}{dt}\right)^2\right)dt = -\int_{x_0}^{x_f}\frac{dV}{dx}\,dx$$

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2\Big]_{t_0}^{t_f} = -V(x)\Big]_{x_0}^{x_f}$$

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2\Big|_{t=t_f} - \frac{m}{2}v_0^2 = V(x_f) - V(x_0)$$

$$m\frac{d^2x}{dt^2} = F(x) = -\frac{dV}{dx}\ ;\ V(x) = -\int F(x)\,dx$$

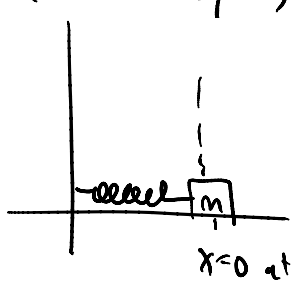
$$x(t_0) = x_0\ ;\ \frac{dx}{dt}\Big|_{t=t_0} = v_0$$

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E_0 \leftarrow \text{Initial energy}$$

$$\frac{m}{2}v_0^2 + V(x_0) =$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E_0 - V(x))}$$ 1st order DEQ, can solve

<u>Ex</u> Mass on a spring obeying Hooke's Law



$x=0$ at equilib

$$m\frac{d^2x}{dt^2} = -kx$$

$$x(0) = x_0$$

$$v(0) = v_0 = \frac{dx}{dt}(0)$$

$$V(x) = -\int F(x)\,dx = k\int x\,dx = k\frac{x^2}{2}$$

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + k\frac{x^2}{2} = E_0$$

$$\frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E_0 - k\frac{x^2}{2}\right)}$$

$$\int \frac{1}{\sqrt{\frac{2E_0}{m} - \frac{k}{m}x^2}}\,dx = \pm\int dt = \pm t + c$$

$$\sqrt{\frac{2E_0}{m} - \frac{k}{m}x^2} = \sqrt{\frac{2E_0}{m}}\sqrt{1 - \frac{k}{2}E_0 x^2}$$

$$x = \sqrt{\frac{2E_0}{k}}\sin\phi$$

$$dx = \sqrt{\frac{2E_0}{k}}\cos\phi\,d\phi$$

$$\sqrt{\frac{m}{2E_0}}\int \frac{1}{\sqrt{1 - \frac{k}{2E_0}x^2}}\,dx = \pm t + c$$

$$1 - \frac{k}{2E_0}x^2 = \sqrt{1 - \frac{k}{2E_0}\frac{2E_0}{k}\sin^2\phi}$$

$$= \sqrt{\cos^2\phi} = \pm\cos\phi$$

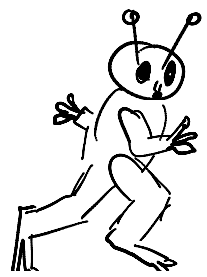
$$\sqrt{\frac{m}{2E_0}}\int \frac{\pm 1}{\cos\phi}\sqrt{\frac{2E_0}{k}}\cos\phi\,d\phi$$

$$\sqrt{\frac{m}{k}}\int \pm\,d\phi = \pm t + c$$

$$\int d\phi = \pm\sqrt{\frac{k}{m}}\,t + \tilde{k}$$

$$\phi = \pm\sqrt{\frac{k}{m}}\,t + \tilde{k}$$

$$x(t) = \sqrt{\frac{2E_0}{k}}\sin\left(\pm\sqrt{\frac{k}{m}}\,t + \tilde{k}\right)$$



$$x(t) = \sqrt{\frac{2E_0}{k}}\sin\left(\pm\sqrt{\frac{k}{m}}\,t + \tilde{k}\right)$$

$$= \pm\sqrt{\frac{2E_0}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t \pm \tilde{k}\right)$$

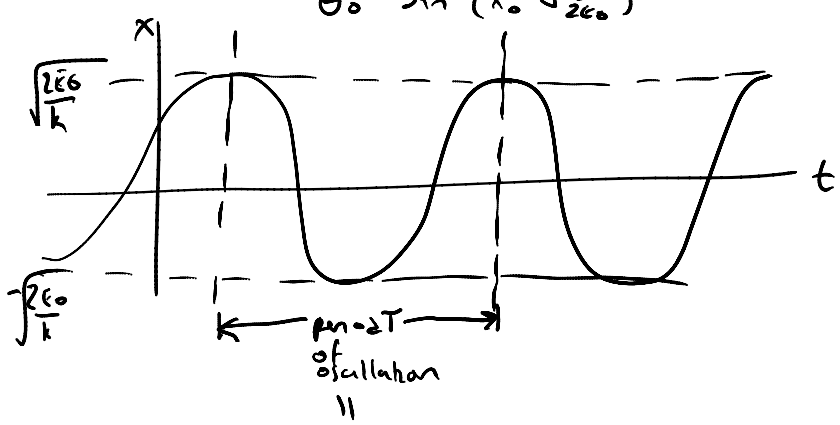
$$\boxed{= \sqrt{\frac{2E_0}{k}}\sin\left(\sqrt{\frac{k}{m}}\,t + \theta_0\right)}$$

if (+), $\tilde{k} = \theta_0$, if (−) $\tilde{k} = -\theta_0$

$$E_0 = \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2$$

$$x(0) = x_0 = \sqrt{\frac{2E_0}{m}}\sin(\theta_0)$$

$$\theta_0 = \sin^{-1}\left(x_0\sqrt{\frac{m}{2E_0}}\right)$$



Amplitude of oscillation ↑ w/ Energy.

Larger oscillations if E is ↑

$$\parallel$$

$$\frac{2\pi}{\sqrt{\frac{k}{m}}} = \sqrt{\frac{m}{k}}\, 2\pi$$

period $\uparrow$ w/ mass $\uparrow$.

$\downarrow$ w/ k

Linear 2nd Order Eqns
Tuesday, April 17, 2007
8:58 AM

Linear 2nd order Eqns

$$y'' + p(x)y' + q(x)y = f(x)$$

Special case where $f(x)=0$, then eqn is <u>homogeneous</u> & linear

If $f(x)\neq 0$, then eqn is linear, <u>inhomogeneous</u>

Linear, homogeneous eqns have a <u>superposition</u> principle

If $y_1(x)$ & $y_2(x)$ solve the eqn (*), then $y(x)=c_1y_1(x)+c_2y_2(x)$ also solves the eqn for any new constants c_1 & c_2

(*) $y''+p(x)y'+q(x)y=0$

$$y' = c_1y_1' + c_2y_2'$$
$$y'' = c_1y_1'' + c_2y_2''$$
$$0 \overset{?}{=} c_1y_1'' + c_2y_2'' + p(x)[c_1y_1' + c_2y_2'] + q(x)(c_1y_1 + c_2y_2)$$

$$c_1\underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{0} + c_2\underbrace{(y_2'' + p(x)y_2' + q(x)y_2)}_{0}$$

b/c y_1, y_2 are solns of the eqn.

$$0=0$$

The general soln will involve 2 constants. In this case, the constants are c_1 & c_2.

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

If you have initial conditions that $y(x_0)=y_0$ & $y'(x_0)=y_1$

$$\Rightarrow c_1y_1(x_0) + c_2y_2(x_0) = y_0$$
$$c_1y_1'(x_0) + c_2y_2'(x_0) = y_1$$

<u>Examples</u>

i) $y''+y=0$

$y_1 = \cos(x)$

$y_2 = \sin(x)$

<u>General soln</u> - $y(x) = c_1\cos(x) + c_2\sin(x)$

$y_3 = 0$ X

$y_1 = \sin(x-\frac{\pi}{4}) = -\cos(x)\sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4})\sin(x)$

$y_2 = \cos(x+\frac{\pi}{4}) = \cos(x)\cos(\frac{\pi}{4}) - \sin(x)\sin(\frac{\pi}{4})$

$\frac{\sqrt{2}}{2}[\cos(x) - \sin(x)]$

$y_2 = -y_1$ X

Easily check if solns are independent or not

We find 2 solns. $y = y_1(x)$ & $y = y_2(x)$

Want to check if they are independent or not

If they are independent then we've found the general soln
$y(x) = c_1 y_1(x) + c_2 y_2(x)$

We obtain a criterion for independence by considering an initial value problem
$y(x_0) = A \quad y'(x_0) = B$

y at $x_0 = \underline{c_1} y_1(x_0) + \underline{c_2} y_2(x_0) = A$
$y'(x_0) = \underline{c_1} y_1'(x_0) + \underline{c_2} y_2'(x_0) = B$

Solve for unknowns

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

2 eqns, 2 unknowns c_1 & c_2

This can be solved, no matter what A & B are if
$$\text{Det} \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \neq 0$$

$$\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$m_3(m_1 x + m_2 y = a)$
$-m_1(m_3 x + m_4 y = b)$

$$m_3 m_2 y - m_1 m_4 y = m_3 a - m_1 b$$
$$y = \frac{m_3 a - m_1 b}{m_3 m_2 - m_1 m_4}$$

So $m_3 m_2 - m_1 m_4 \neq 0$
Det.

Wronskian
$$W(x) = \text{Det} \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$$

$= y_1(x)y_2'(x) - y_1'(x)y_2(x)$

$W = 0 \Rightarrow$ solns not independent

$y'' + y = 0$

$y_1 = \cos(x)$

$y_2 = \sin(x)$

$\cancel{y_3 = 0}$

$y_4 = \sin(x - \frac{\pi}{4})$

$y_5 = \cos(x + \frac{\pi}{4})$

$\{\cos(x), 0\}$

$W = \det\begin{bmatrix} \cos(x) & 0 \\ -\sin(x) & 0 \end{bmatrix}$

$W = 0$

So that doesn't work

$\{\cos(x), \sin(x)\}$

$W = \det\begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

$= \cos^2(x) + \sin^2(x) = 1$

✓

$\{\sin(x - \frac{\pi}{4}), \cos(x + \frac{\pi}{4})\}$

$W = \det\begin{bmatrix} \sin(x - \frac{\pi}{4}) & \cos(x + \frac{\pi}{4}) \\ \cos(x - \frac{\pi}{4}) & -\sin(x + \frac{\pi}{4}) \end{bmatrix}$

$= -\sin(x - \frac{\pi}{4})\sin(x + \frac{\pi}{4}) - \cos(x - \frac{\pi}{4})\cos(x + \frac{\pi}{4})$

X

$= -\cos(x - \frac{\pi}{4} - x - \frac{\pi}{4})$

$\cos(\frac{\pi}{2}) = 0$

If the Wronskian $= 0$ at any pt it $= 0$ everywhere

The following statements are equivalent

1) y_1 & y_2 form a fundamental set of solns (ie the gen soln

2) y_1 & y_2 are linearly independent. $y_2 \neq c y_1$ is $c_1 y_1 + c_2 y_2$

3) Wronskian $W(y_1, y_2) = y_1 y_2' + y_1' y_2 \neq 0$ at one pt $x = x_0$

4) $W(y_1, y_2) \neq 0$ for all x

Linear 2nd order Homogenous ODE's

Wednesday, April 18, 2007
8:58 AM

<u>Recap</u>

Linear, 2nd order homogeneous ODE's

$$(*) \quad y'' + p(x)y' + q(x)y = 0$$

There is a superposition principle. If $y_1(x)$ & $y_2(x)$ solve $(*)$ then so does $c_1y_1(x) + c_2y_2(x)$ for any constants c_1 & c_2

Need 2 linearly ind. solns y_1 & y_2 to get the general soln $y = c_1y_2 + c_2y_2$;

- linearly independent means $y_2 \neq Cy_1$ for any constant C
- y_1 & y_2 are a "fundamental set"

Easy check of linear independence

Compute Wronskian $W(y_1, y_2) = \text{Det}\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}$

If $W(y_1, y_2) = 0$ then y_1 & y_2 are lin. <u>dependent</u>

If $W(y_1, y_2) \neq 0$ then y_1 & y_2 are lin <u>independent</u>

It's enough to check W at one pt. If $W(y_1(x_0), y_2(x_0)) \neq 0$ then solns are lin. independent

if $W(y_1(x_0), y_2(x_0)) = 0$ then $W(y_1(x), y_2(x)) = 0$

Special case where p & q are constants

$$ay'' + by' + cy = 0$$

$a \neq 0$

a, b, c are constants

linear 2nd order homogeneous constant coefficient eqn

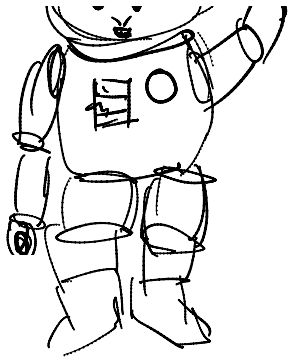
<u>Method:</u> try $y = y_0 e^{rx}$

$$y' = ry_0e^{rx} = ry$$
$$y'' = r^2y_0e^{rx} = r^2y$$

$$ay'' + by' + cy = 0$$
$$ar^2y + bry + cy = 0$$

$\Rightarrow$ Need r to satisfy $\boxed{ar^2 + br + c = 0}$

"Characteristic eqn"



$y = y_0 e^{ry}$

$ar^2 + br + c = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

① $b^2 - 4ac > 0$ $\quad r_1$ & r_2 are real

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ 2 solns, distinct + real $\Rightarrow r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

② $b^2 - 4ac = 0$

one soln, $r_1 = \frac{-b}{2a}$ $\}$ $y_1 = e^{r_1 x}$; $y_2 = ?$

③ $b^2 - 4ac < 0$

2 complex solns $r_1 = -\frac{b}{2a} + \frac{i\sqrt{4ac - b^2}}{2a}$

$r_2 = -\frac{b}{2a} - \frac{i\sqrt{4ac - b^2}}{2a}$

<u>Case 1</u> 2 solns

<u>Ex1</u> $2y'' - 3y' + y = 0$

$2r^2 - 3r + 1 = 0$

$(2r - 1)(r - 1) = 0$

$r = \frac{1}{2}, r = 1$

$2r^2 - 3r + 1$

$(2r - 1)(r - 1)$

-1

-2

$y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{x}$

$y = e^{rx}$

Check that $y_1 = e^{r_1 x}$ $\quad y_2 = e^{r_2 x}$ are lin ind

$r_1 \neq r_2$

$$W(y_1, y_2) = \text{Det}\begin{bmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{bmatrix}$$

$x = 0$

$$\text{Det}\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} = r_2 - r_1 \neq 0$$

<u>Ex2</u> $y = e^{rx}$

$y'' + 8y' - 9y = 0$

$y(0) = 1$

$y'(0) = 0$

$r^2 + 8r - 9 = 0$

$(r + 9)(r - 1) = 0$

$r = 1, -9$

$y = c_1 e^{-9x} + c_2 e^{x}$

$y' = -9c_1 e^{-9x} + c_2 e^{x}$

$y = c_1 e^{\quad} + c_2 e^{\quad}$

$y(0) = c_1 + c_2 = 1$

$y'(0) = -9c_1 + c_2 = 0 \Rightarrow c_2 = 9c_1$

$c_1 + 9c_1 = 1$

$c_1 = \frac{1}{10}$

$c_2 = \frac{9}{10}$

$$y = \frac{1}{10} e^{-9x} + \frac{9}{10} e^{x}$$

<u>Case 2</u> 1 soln

$b^2 - 4ac = 0$

$r_1 = -\frac{b}{2a}$

$y_1 = e^{-\frac{b}{2a}x}$ $\underline{\text{try}}$ $y_2 = v(x) e^{-\frac{b}{2a}x}$

$y = v(x) e^{-\frac{b}{2a}x}$

$y' = v'(x) e^{-\frac{b}{2a}x} - v(x) \frac{b}{2a} e^{-\frac{b}{2a}x}$

$= (v' - \frac{b}{2a} v) e^{-\frac{b}{2a}x}$

$y'' = (v'' - \frac{b}{2a} v') e^{-\frac{b}{2a}x} +$ same method

$(v' - \frac{b}{2a} v)(-\frac{b}{2a} e^{-\frac{b}{2a}x})$

$= (v'' - \frac{b}{a} v' + \frac{b^2}{4a} v) e^{-\frac{b}{2a}x}$

$ay'' + by' + cy = e^{-\frac{b}{2a}x} [av'' - bv' + \frac{b^2}{4a} v + bv' - \frac{b^2}{2a} v + cv]$

$= 0$

$= e^{-\frac{b}{2a}} [av'' + \underset{\substack{\uparrow \\ \text{since} \\ \frac{b^2}{4a} = c}}{c} v - 2c v + cv] = 0$

$= e^{-\frac{b}{2a}} [av''] = 0$

so $v'' = 0$

$\Rightarrow v'' = 0$

$\Rightarrow v = \alpha + \beta x$

$y_2 = (\alpha + \beta x) e^{-\frac{b}{2a}x}$

$y_2 = \alpha y_1 + x \beta y_1$

set $\alpha = 0$

If $b^2 - 4ac = 0$, then 1 soln is

$y_1 = e^{-\frac{b}{2a}x}$

$y_2 = x e^{-\frac{b}{2a}x}$

<u>Ex 1</u>

$y'' - 2y' + y = 0$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1 \rightarrow y_1 = e^x$$

$$y_2 = xe^x$$

gen soln $\boxed{y = c_1 e^x + c_2 x e^x}$

2nd Order linear Homogenous Const Coefficient Diff Eqns

Friday, April 20, 2007
9:04 AM

$ay'' + by' + cy = 0 \quad ; \quad a \neq 0$

2nd order linear homogeneous constant coefficient diff eqn

$y = e^{rx} \; ; \; y' = re^{rx} \; ; \; y'' = r^2 e^{rx}$

$e^{rx}[ar^2 + br + c] = 0$

$\Rightarrow r$ solves $ar^2 + br + c = 0$ $\Leftarrow$ characteristic eqn

$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

① $b^2 - 4ac > 0$
② $b^2 - 4ac = 0$
③ $b^2 - 4ac < 0$

① if $b^2 - 4ac > 0$
2 real roots that are distinct

$r_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$

$r_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

② $b^2 - 4ac = 0$

$r = -\frac{b}{2a}$

$y = c_1 e^{rx} + c_2 x e^{rx}$

<u>Ex.</u> $y'' + 4y' + 4y = 0$

$y(0) = 0$

$y'(0) = 1$

Beher
Professional Patent

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0$

$r = -2$

$y = c_1 e^{-2x} + c_2 x e^{-2x} \qquad y(0) = c_1 = 0$

$y' = c_2 e^{-2x} - 2c_2 x e^{-2x} \qquad y'(0) \quad c_2 = 1$

$=$

$$\begin{cases} y = xe^{-2x} \\ y' = e^{-2x} - 2xe^{-2x} \\ y'' = -2e^{-2x} - 2e^{-2x} + 4xe^{-2x} = -4e^{-2x} + 4xe^{-2x} \end{cases}$$

$y'' + 4y' + 4y$

$= -4e^{-2x} + 4xe^{-2x}$

$+ 4e^{-2x} - 8xe^{-2x}$

$+ 4xe^{-2x} = 0$

$+4xe^{-2x}=0$

if $b^2-4ac<0$

2 complex roots $\quad r=\alpha\pm i\beta$

$$\left.\begin{array}{l} r_1=\alpha+i\beta \\ r_2=\alpha-i\beta \end{array}\right\}\text{"complex conjugate pairs"}$$

$$r=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=-\frac{b}{2a}\pm\frac{\sqrt{(-1)(4ac-b^2)}}{2a}$$

$$=-\frac{b}{2a}\pm\frac{i\sqrt{4ac-b^2}}{2a}$$

$$\alpha=-\frac{b}{2a}\qquad \beta=\frac{\sqrt{4ac-b^2}}{2a}$$

$$y_1=e^{(\alpha+i\beta)x}\qquad y_2=e^{(\alpha-i\beta)x}$$

$$=e^{\alpha x}e^{i\beta x}\qquad =e^{i\alpha x}e^{-i\beta x}$$

$$=e^{\alpha x}[\cos(\beta x)+i\sin(\beta x)]\qquad e^{\alpha x}[\cos(\beta x)-i\sin(\beta x)]$$

Euler's Formula

$$e^{i\theta}=\cos\theta+i\sin\theta$$

$$e^{\pm i\theta}=\cos\theta\pm i\sin\theta$$

$$u_1=\tfrac{1}{2}(y_1+y_2)$$

$$=e^{\alpha x}\cos(\beta x)$$

$$u_2=\frac{1}{2i}(y_1-y_2)$$

$$=e^{\alpha x}(\sin(\beta x))$$

$$\text{Det}\begin{bmatrix} c_1e^{\alpha x}\cos\beta x & c_2e^{\alpha x}\sin\beta x \\ c_1\alpha xe^{\alpha x}\cos\beta x-c_1e^{\alpha x}\sin\beta x & \text{same thing} \end{bmatrix}$$

① $b^2-4ac>0$ $\quad y=c_1e^{r_1x}+c_2e^{r_2x}$

② $b^2-4ac=0$ $\quad y=c_1e^{rx}+xc_2e^{rx}$

③ $b^2-4ac<0$ $\quad c_1e^{\alpha x}\cos\beta x+c_2e^{\alpha x}\sin(\beta x)$

$\alpha=Re(r)$ real

$\beta=Im(\beta)$ imaginary

Use Taylor's series to motive Euler's Formula.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}\cdots$$

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\frac{(i\theta)^6}{6!}\cdots$$

$i^2=-1\quad i^3=-i\quad i^4=1\quad i^5=i\quad i^6=-1$

$$e^{i\theta} = 1 + i\theta + -\frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \cdots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots\right)}_{\sin\theta}$$

$$z = \cos\theta + i\sin\theta$$

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$$

$$\frac{dz}{d\theta} = i(i\sin\theta + \cos\theta) = iz$$

$$\frac{dz}{d\theta} = iz \Rightarrow z = ce^{i\theta}$$

<u>Ex</u>

$$u_1 = e^{\alpha x}\cos(\beta x) \qquad W(u_1, u_2) = \mathrm{Det}\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix}$$

$$u_2 = e^{\alpha x}\sin(\beta x)$$

$$\mathrm{Det}\begin{bmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x) & \alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x) \end{bmatrix}$$

$$W(u_1, u_2)\Big|_{x=0} = \mathrm{Det}\begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix} = \beta$$

β cannot be 0 b/c that would be a diff case.

<u>Ex</u> $y'' + 2y' + 5y = 0$

$$y(0) = 0$$
$$y'(0) = 1$$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2a} = \frac{-2 \pm 4i}{2a} = -1 \pm 2i$$

$$y = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)$$

$$y(0) = c_1 = 0 \Rightarrow y = c_2 e^{-x}\sin(2x)$$

$$y' = c_2 e^{-x}\sin(2x) + 2c_2 e^{-x}\cos(2x)$$

$$y'(0) = 2c_2 = 1$$

Soln

Soln

$y = \frac{1}{2}e^{-x}\sin(2x)$

$y = a_1 e^{(-1+2i)x} + a_2 e^{(-1-2i)x}$

Linear nth order homogenous diff eqns

Monday, April 23, 2007
8:59 AM

Linear n^{th} order homogeneous differential eqns

$$y^{(n)} = \frac{d^{(n)}y}{dx^n} = n^{th} \text{ derivative}$$

$$(*) \quad y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \dots + p_{n-1}(x)y' + p_n(x)y = 0$$

There is a superposition principle

If $y_1(x)$ & $y_2(x)$ solve $(*)$ then so does

$c_1 y(x) + c_2 y_2(x) = y$ for any real constants c_1 & c_2

We can expect n constants of integration for n^{th} order eqn.

Unique sol. requires n initial conditions

General soln of $(*)$ is $y(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$
where $y_1, y_2, y_3 \dots y_n$ are n linearly ind. solns of $(*)$

Solns $y_1, \dots y_n$ are linearly independent if the only sol. to $c_1 y_1 + c_2 y_2 + c_3 y_3 \dots + c_n y_n = 0$ is to choose $c_1, c_2, c_3 \dots c_n = 0$

Otherwise one of the solns could be expressed as a linear combination of the others.

$n \times n$ matrix

We test for linear independence of a set of functions $\{y_1, y_2 \dots y_n\}$ by computing the Wronskian

$$W(y_1 \dots y_n) = \text{Det}\begin{bmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ y_1' & y_2' & y_3' & \dots & y_n' \\ \vdots & & & & \vdots \\ y_1^{n-1} & y_2^{n-1} & y_3^{n-1} & \dots & y_n^{n-1} \end{bmatrix}$$

Must get a square matrix

if $W \neq 0 \Rightarrow y_1 \dots y_n$ are linearly <u>dependent</u>

if $W \neq 0 \Rightarrow y_1 \dots y_n$ are linearly <u>independent</u>

Can check w/ just 1 pt.

<u>Problem</u>: Use the Wronskian to show that $y'' + p(x)y' + q(x)y = 0$ cannot have 3 lin. ind. solns.

y_1, y_2, y_3

$$W(y_1, y_2, y_3) = \text{Det}\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$$

$$= y_1(y_2'y_3'' - y_3'y_2'') - y_2(y_1'y_3'' - y_3'y_1'') + y_3(y_1'y_2'' - y_2'y_1'')$$

$$= y_1(y_2'(-py_3' - qy_3) - y_2'(-py_2' - qy_2)) - y_2(y_1'(-py_3' - qy_3) - y_3'(-py_1' - qy_1)) + y_3(y_1'(-py_2' - qy_2) - y_2'(-py_1' - qy_1))$$

$$= 0$$

Diff eqns w/ const. coefficients

$$a_n y^{(n)} + a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \dots + a_1 y' + a_0 y = 0$$

Assume soln in the form of $e^{rx} = y$

$$e^{rx}\underbrace{[r^n a_n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0]}_{\text{characteristic eqn that } r \text{ must satisfy}} = 0$$

<u>Ex</u> If the characteristic eqn is $(r-1)(r-2)^3(r^2+4)^2 = 0$ what is the general soln?

<u>Roots</u> 8th order, so 8 roots.

$\{1, 2, 2, 2, 2i, -2i, 2i, -2i\}$

Prescription for going from roots of the char eqn to the gen soln

(1) n Real, distinct roots $\{r_1, \dots r_n\}$, then we have n linearly ind. solns

$e^{r_1 x}, e^{r_2 x}, \dots e^{r_n x}$

(2) Any real root r, that is repeated k times ($k = 2$ or 3 or $\dots$ n) then k solns associated w/ that root r are $e^{rx}, xe^{rx}, x^2e^{rx}, \dots x^{k-1}e^{rx}$

(3) Distinct complex pair $r = \alpha + i\beta$ & $r = \alpha - i\beta \Rightarrow$ 2 lin ind solns. $e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$

(4) Repeated complex roots $r = \alpha + i\beta$, $r = \alpha - i\beta$ repeated k

times. Same thing.

$e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$,
$xe^{\alpha x}\cos(\beta x)$, $xe^{\alpha x}\sin(\beta x)$,

$\vdots$

$x^{k-1}e^{\alpha x}\cos(\beta x)$, $x^{k-1}e^{\alpha x}\sin(\beta x)$

$$y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 x e^{2x} + c_4 x^2 e^{2x} + c_5\cos(2x) + c_6\sin(2x) + c_7\, x\cos(2x) + c_8\, x\sin(2x)$$

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

$a_n \neq 0$

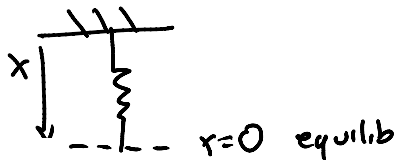
(*) $a_n, a_{n-1}, \ldots, a_0$ are real

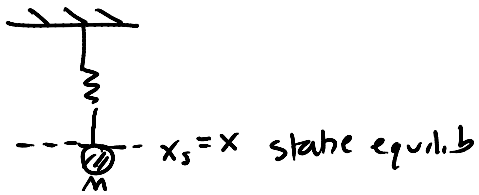
Claim if $r = \alpha + i\beta$ $(\beta \neq 0)$ is a soln of (*) then so is $r = \alpha - i\beta$

$$a_n(r-r_1)(r-r_2)\cdots(r-r_n) = 0$$

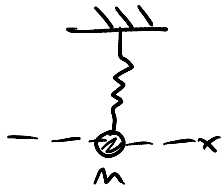
Spring Constant

Tuesday, April 24, 2007
8:58 AM





k = spring constant



$\vec{F} = m\vec{a}$ $\quad \vec{a} = \frac{d^2x}{dt^2}\hat{i} = \ddot{x}\,\hat{i}$

dots used when indep. var. is t

$\vec{F} = \vec{F}_{grav} + \vec{F}_{spring} + \vec{F}_{damper}$

$= \hat{i}\left[mg - kx - b\dot{x}\right]$

$m\ddot{x} = mg - kx - b\dot{x}$ $\quad$ 2nd order lin eqn. Not homogeneous b/c of mg

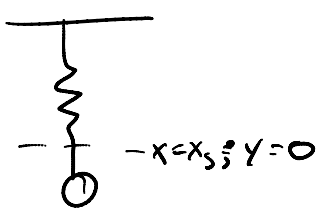
$m\ddot{x} + kx + b\dot{x} = mg$

Introduce a new coordinate system where $y = x - x_s$

$x = x_s + y$

x_s : $\dot{x} = \ddot{x} = 0$

$kx_s = mg$ $\qquad y = x - \frac{mg}{k}$



$\dot{y} = \dot{x}$

$\ddot{y} = \ddot{x}$

$m\ddot{y} + b\dot{y} + k\left(y + \frac{mg}{k}\right) = mg$

$m\ddot{y} + b\dot{y} + ky = 0$

Try $e^{rt} = y$ $\qquad mr^2 + br + k = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4(m)(k)}}{2m}$

(I) Undamped Case $b = 0$

$r = \frac{\pm\sqrt{-4mk}}{2m} = \frac{\pm 2i\sqrt{mk}}{2m} = \pm i\frac{\sqrt{mk}}{m} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$

$y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$

or equivalently $y = A\cos(\omega_0 t - \phi)$

$\{c_1, c_2\}$ or $\{A, \phi\}$

$$\cos(A-B) = \mathrm{Re}\left[e^{i(A-B)}\right] = \mathrm{Re}\left[e^{iA}e^{-iB}\right]$$

"real part"

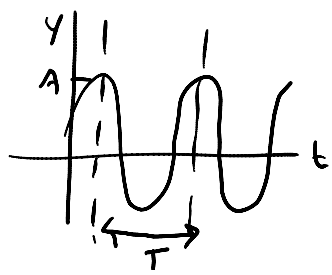
$$\mathrm{Re}\left[(\cos A + i\sin A)(\cos B - i\sin B)\right]$$
$$= \cos A\cos B + \sin A \sin B$$

$$A\cos(\omega_0 t)\cos(\phi) + A\sin(\omega_0 t)\sin(\phi)$$

$$A\cos(\phi) = c_1 \qquad A\sin(\phi) = c_2$$

$$A = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \tan^{-1}\left(\frac{c_2}{c_1}\right) \quad \text{b/c} \quad \frac{A\sin\phi}{A\cos\phi} = \frac{c_2}{c_1}$$



$$T = \frac{2\pi}{\omega_0}$$

(II) Damped Case

(A) $b^2 - 4mk < 0$ b/c b is very small damped vibration

(B) $b^2 = 4mk$ "critically damped"

(C) $b^2 - 4mk > 0$ overdamped case

(A) $r = -\frac{b}{2m} \pm i\sqrt{\frac{4mk - b^2}{4m^2}}$

$$= -\frac{b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad \gamma = \frac{b}{2m} \qquad \omega_0 = \pm\sqrt{\frac{k}{m}}$$

$$= -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$$

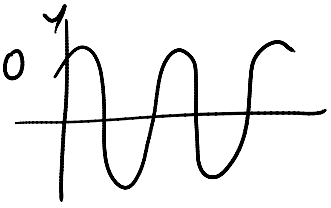
$$= c_1 e^{-\gamma t}\cos(\omega t) + c_2 e^{-\gamma t}\sin(\omega t)$$

$$= Ae^{\gamma t}\cos(\omega t - \phi)$$

$$\omega^2 = \omega_0^2 - \gamma^2$$

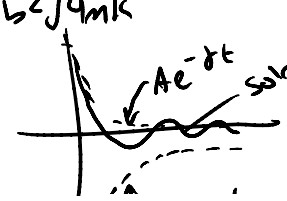
$$\omega^2 < \omega_0^2$$

$b = 0$

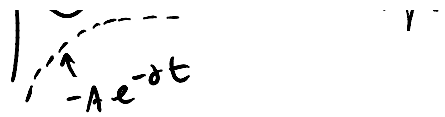


$$y = A\cos(\omega_0 t - \phi)$$

$0 < b < \sqrt{4mk}$



$$y = \underbrace{Ae^{-\gamma t}}_{\text{amp. of cos function}}\cos(\omega t - \phi)$$



(B) $b = \sqrt{4mk}$ $\qquad y = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t}$

$$= e^{-\gamma t}(c_1 + c_2 t)$$

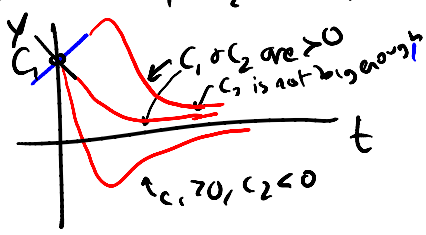
if c_1 & c_2 are both pos. then $y(t)$ never goes through $y=0$

if $c_1 > 0$ & $c_2 < 0$ then $y(t) = 0$ when $t = -\frac{c_1}{c_2}$

$y = e^{-\gamma t}(c_1 + c_2 t) \qquad y' = -\gamma e^{-\gamma t}(c_1 + c_2 t) + c_2 e^{-\gamma t} = \dot{y}$

$y(t=0) = c_1$ = initial displacement

$\dot{y}(0) = -\gamma c_1 + c_2$ = initial velocity



if $c_1 > 0$, then graph depends on c_2

Linear Non-Homogenous Eqns
Wednesday, April 25, 2007
9:02 AM

Linear Non-Homogeneous eqns

eg $y'' + p(x)y' + q(x)y = f(x) \neq 0$ $*$

[$f(x) = 0 \Rightarrow$ homogeneous]

Introduce as shorthand linear operator L

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

$$Ly = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y$$

Eq $*$ is then written $Ly = f(x)$

<u>Observation 1</u>

$y = 0$ is not a soln of $*$

<u>Observation 2</u>

Eqn $*$ does not have a superposition principle
If y_1 & y_2 solve $*$ then $y = c_1y_1 + c_2y_2$ does not necessarily solve it.

$Ly_1 = f(x)$
$Ly_2 = f(x)$

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2$$
$$= (c_1 + c_2)f \neq f$$

If $c_1 + c_2 = 1$, then it works. Otherwise no.

<u>Observation 3</u>

If y_1 & y_2 solve $*$ then it must be that
$y_2 = y_1 + y_h$, where y_h is a soln of the homogeneous problem associated w/ $*$.

$Ly_h = 0$

$Ly = f(x)$

$Ly_1 = f(x)$
$Ly_2 = f(x)$

$$L(y_2 - y_1) = Ly_2 - Ly_1 = f - f = 0$$

$$L(y_2 - y_1) = 0$$

* solves the homogeneous problem.

Strategy for finding the general soln of *

① Find a particular soln $y = y_p$ $(Ly_p = f(x))$

② Solve the associated homogeneous problem $(Ly_h = 0)$ (gen soln)

③ Gen soln is $y = y_p + y_h$

<u>Ex 1</u>

$$y'' - y = e^{2x}$$

Look for particular soln $y_p = ce^{2x}$
$y_p'' = 4ce^{2x}$

$$y_p'' - y_p = 4ce^{2x} - ce^{2x} = 3ce^{2x} = e^{2x}, \boxed{c = \tfrac{1}{3}} \checkmark$$

① $y = \frac{1}{3}e^{2x}$

$y_p'' - y_h = 0$
$r^2 - 1 = 0$
$r = \pm 1$

② $y_h = c_1 e^x + c_2 e^{-x}$

③ $\boxed{\frac{1}{3}e^{2x} + c_1 e^x + c_2 e^{-x} = y}$

<u>Ex 2</u>

$$y'' + y = x^2$$

Look for particular soln
$y_p = x^2 + Ax + B$

$$y_p'' = 2$$

$$y_p'' + y_p = x^2 + Ax + Br - 2 = x^2$$

$$A = 0 \qquad B = -2$$

① $y_p = x^2 - 2$

$$y_h'' + y_h = 0$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

$$y = x^2 - 2 + c_1 \cos(x) + c_2 \sin(x)$$

<u>Ex3</u>

$$y'' - 2y' - y = \sin(x)$$

$$y_p = A\cos(x) + B\sin(x)$$

$$y_p' = -A\sin(x) + B\cos(x)$$

$$y_p'' = -A\cos(x) - B\sin(x)$$

$$\sin(x) = -A\cos(x) - B\sin(x) + 2A\sin(x) - 2B\cos(x) - A\cos(x) - B\sin(x)$$

$$\sin(x) = \sin(x)\,[-B + 2A - B]$$

$$+\cos(x)\,[-A - 2B - A]$$

$$2(A - B) = 1$$

$$-2(A + B) = 0$$

$$B = -A$$

$$A = \frac{1}{4}$$

$$B = -\frac{1}{4}$$

① Particular soln $\frac{1}{4}(\cos(x) - \sin(x))$

$$y_h'' - 2y_h' - y_h = 0$$

$$r^2 - 2r - 1 = 0$$

$$r = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

② Homogeneous soln

$$y_h = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$$

$$y = y_p + y_h$$

<u>Method of Undetermined Coefficients</u>

Works when $f(x)$ involves functions like $e^{\alpha x}$, $\cos(\beta x)$, $\sin(\beta x)$, polynomials, & combinations of these.

Solving * w/ const. coefficients

e.g. $e^{\alpha x}\cos(\beta x)$ or $x^3 e^x$

<u>Ex 4</u>

$$y'' + ay' + by = e^x$$

In what instance does $y_p \neq ce^x$?

$$y'' + 2y' - y = x^3$$

$$y_p = \underset{\substack{\uparrow \\ (-1)}}{A}x^3 + Bx^2 + Cx + D$$

y

Examples

Tuesday, May 01, 2007
9:07 AM

Ex1 $y'' + 2y' + y = xe^{2x}$ Ex2 $y'' - y' = e^x$ Ex3 $y'' - y' = x^2 e^x$

Ex4 $y'' + 2y' + y = e^{-x}$ Ex5 $y'' + y = \sin x$

Ex1 $y'' + 2y' + y = xe^{2x}$

$$y_p = Axe^{2x} + Be^{2x}$$
$$y_p' = 2Axe^{2x} + Ae^{2x} + 2Be^{2x} = (A+2B)e^{2x} + 2Axe^{2x}$$
$$y_p'' = 2(A+2B)e^{2x} + 2Ae^{2x} + 4Axe^{2x} = (4A+4B)e^{2x} + 4Axe^{2x}$$

$6A + 9B = 0$ if eqn =

$$y_p'' + 2y_p' + y_p = e^{2x}[4A + 4B + 2A + 4B + B] + xe^{2x}[4A + 4A + A] = xe^{2x}$$

$9A = 1$ if eqn =

$$A = \frac{1}{9}$$
$$B = -\frac{2}{27}$$

$$\boxed{y_p = \frac{1}{9}xe^{2x} - \frac{2}{27}e^{2x}}$$

$r^2 + 2r + 1 = 0 = (r+1)^2$

homogeneous soln $= y = c_1 e^{-x} + c_2 x e^{-x}$

$$y = y_p + y_h = \frac{1}{9}xe^{2x} - \frac{2}{27}e^{2x} + c_1 e^{-x} + c_2 x e^{-x}$$

Ex2 $y'' - y' = e^x$

$$y_p = Ae^x$$
$$y_p' = Ae^x$$
$$y_p'' = Ae^x$$
$$Ae^x - Ae^x = e^x$$
$$0 = e^x$$

So there is no way to choose A to get e^x out.

$$y'' - y' = 0$$
$$r^2 - r = 0$$
$$r^2 = r$$
$$r = 1, 0$$

$y_h = c_1 + c_2 e^x$ this is a soln, & we used a guess like this so it doesn't work.

$$u' - u = e^x$$

$$\mu = e^{\int -dx} = e^{-x}$$

$$e^{-x}u' - e^{-x}u = 1$$

$$\int \frac{d}{dx}(e^{-x}u)\,dx = \int 1\,dx$$

$$e^x u = x + C$$
$$u = xe^x + Ce^x$$
$$y' = xe^x + Ce^x$$
$$y = \int (xe^x + Ce^x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$
$$\int xe^x\,dx$$
$$u = x \qquad du = dx$$
$$dv = e^x dx \qquad v = e^x$$
$$\int xe^x\,dx = xe^x - \int e^x dx$$
$$= xe^x - e^x + B$$

$$y = xe^x - e^x + B + Ce^x$$
$$= xe^x + c_1 + c_2 e^x$$

<u>Advice</u>

First determine y_h, so you will know what to guess for y_p

<u>Exception #1</u> if the forcing term $f(x)$ (right-hand side) contains an exponential or a trig function or a combo of the 2 that comes from a simple root (not repeated) of the characteristic eqn then one must multiply the trial function (your guess) for y_p we'd normally use by x.

<u>Ex. 2</u>

① Find homogeneous soln

$$y_h'' - y_h' = 0$$
$$r^2 - r = 0$$
$$r(r-1) = 0$$
$$y_h = c_1 + c_2 e^x$$

② See e^x & change to xe^x

$$y_p = Axe^x$$
$$y_p' = Ae^x + Axe^x$$
$$y_p'' = Ae^x + Ae^x + Axe^x$$
$$= 2Ae^x + Axe^x$$
$$y_p'' - y_p' = 2Ae^x + Axe^x - Ae^x - Axe^x = Ae^x \Rightarrow A = 1$$

$$y = y_h + y_p = c_1 + c_2 e^x + xe^x$$

<u>Ex 3</u>. $y'' - y' = x^2 e^x$

① $y_h'' - y_h' = 0$
$$y_h = c_1 + c_2 e^x$$

② $y_p = x(Ax^2e^x + Bxe^x + \underline{Ce^x})$ ← [illegible]
$$y_p = Ax^3e^x + Bx^2e^x + Cxe^x$$
$$y_p' = 3Ax^2e^x + Ax^3e^x + 2Bxe^x + Bx^2e^x + Ce^x + Cxe^x$$
$$y_p'' = y_p' + 3Ax^2e^x + 2Bxe^x + Ce^x + 6Axe^x + 2Be^x$$
$$y_p'' - y_p' = 3Ax^2e^x + (2B + 6A)xe^x + (C + 2B)e^x$$

$3A$ must be 1

$2B+6A$ must be 0

$C+2B=0$

$A=\frac{1}{3}$

$B=-1$

$C=2$

$$y=C_1+C_2e^x+\frac{1}{3}x^3e^x-x^2e^x+2xe^x$$

Oscillations

Wednesday, May 02, 2007
9:03 AM

$y'' + ay' + by = f(x)$ Lab due Mon May 14

$y = y_h + y_p$

y_p = "particular soln"
y_h = gen soln of homogeneous prob

$$y_h'' + ay_h' + by_h = 0$$

Step 1 Find y_h
Step 2 Check to see relationship btwn y_h & guess y_p
If not, then "guess" y_p w/ undetermined coeff's & solve for these

Exception 1 If yes, then modify guess by multiplying it by x if the root is simple

Exception 2 If the forcing $f(x)$ contains an exponential that comes from a repeated root of the characteristic eqn then multiply the trial function for y_p by x^2

Ex

$y'' + 2y' + y = e^{-x}$ $y_c = y_h$ in the book

① $y_h'' + 2y_h' + y_h = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1 \Rightarrow$ repeated root

$y_h = c_1 e^{-x} + c_2 x e^{-x}$

② $y_p = (Ae^{-x})x^2 = Ax^2e^{-x}$

$y_p' = 2Axe^{-x} - Ax^2e^{-x}$

$y_p'' = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$

$y_p'' = 2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x}$

$y_p'' + 2y_p' + y_p = e^{-x}$

$= e^{-x}[2A]$
$+ xe^{-x}[-4A + 4A]$
$+ x^2e^{-x}[A - 2A + A] = 2Ae^{-x} = e^{-x}$

$A = \frac{1}{2}$

$y = \frac{1}{2}x^2e^{-x} + c_1e^{-x} + c_2xe^{-x}$

$y = \frac{1}{2}x^2$

$\ddot{x} = \frac{d^2x}{dt^2}$

$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$

Forced Mechanical Vibration
-resonance

Homogeneous soln

$m\ddot{x}_h + b\dot{x}_h + kx_h = 0$

$mr^2 + br + k = 0$

$\frac{-b \pm \sqrt{b^2 - 4(m)(k)}}{2m} = r$

$-\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$

Focus on decaying oscillations when not much damping

$\left(\frac{b}{2m} << \sqrt{\frac{k}{m}}\right)$

$\omega_0 = \sqrt{\frac{k}{m}}$ = frequency of oscillations in undamped prob

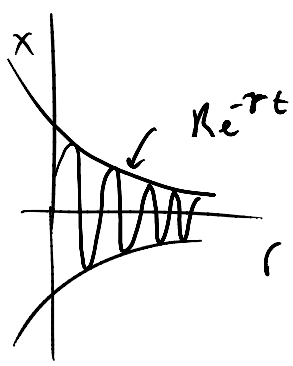
$\gamma = \frac{b}{2m}$ = decay rate associated w/ damping

$-\gamma \pm \sqrt{\omega_0^2 + \gamma^2}$

$x_h = c_1 e^{-x} \cos\left(\left(\sqrt{\omega_0^2 - \gamma^2}\right)t\right) + c_2 e^{-\gamma t} \sin\left(\left(\sqrt{\omega_0^2 - \gamma^2}\right)t\right)$

damping causes shift to lower freq.

$\sqrt{\omega_0^2 - \gamma^2} < \omega_0$



$x = Re^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2}\, t + \phi\right)$

$m\ddot{x}_p + b\dot{x}_p + kx = F_0 \cos(\omega t)$

unless both $\gamma = 0$ & $\omega = \omega_0$ the forcing term is not one of the homogeneous solns

$x_p = A\cos(\omega t) + B\sin(\omega t)$

$\dot{x}_p = -\omega A \sin(\omega t) + \omega B \cos(\omega t)$

$\ddot{x}_p = -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)$

$$m\ddot{x}_p + b\dot{x}_p + kx_p = \cos(\omega t)\overbrace{[-m\omega^2 A + b\omega B + kA]}^{F_0} + \sin(\omega t)\underbrace{[-m\omega^2 B - b\omega A + kA]}_{0} = F_0\cos(\omega t)$$

$$\left.\begin{aligned} \tfrac{1}{m}\left[(-m\omega^2+k)A + (b\omega)B = F_0\right] \\ \tfrac{1}{m}\left[(-m\omega^2+k)B - (b\omega)A = 0\right] \end{aligned}\right\} \begin{aligned} (\omega_0^2-\omega^2)A + (2\gamma\omega)B &= \tfrac{F_0}{m} \\ (\omega_0^2-\omega^2)B - (2\gamma\omega)A &= 0 \end{aligned}$$

$$\left[\frac{(\omega_0^2-\omega^2)^2}{2\omega\gamma} + \frac{(2\gamma\omega)^2}{2\gamma\omega}\right]B = \frac{F_0}{m}$$

$$B = \frac{2\gamma\omega F_0}{m[(\omega_0^2-\omega^2)^2 + (2\gamma\omega)^2]}$$

$$A = \frac{(\omega_0^2-\omega^2)F_0}{m[(\omega_0^2-\omega_0^2)^2 + (2\gamma\omega)^2]}$$

$$x_p = \frac{F_0}{m[(\omega_0^2-\omega^2)^2 + (2\gamma\omega)^2]}\left[(\omega_0^2-\omega^2)\cos(\omega t) + 2\gamma\omega\sin(\omega t)\right]$$

$$= R\cos(\omega t + \phi)$$

$$C = \sqrt{c_1^2 + c_2^2}$$

$$R^2 = c_1 + c_2$$

$$\frac{F_0}{m}\sqrt{(\omega_0^2-\omega^2)^2 + (2\gamma\omega)^2}$$

(all that stuff)

$$R = \frac{F_0}{m}\left(\sqrt{(\omega_0^2-\omega^2)^2 + (2\gamma\omega)^2}\right)^{-1}$$

Denominator is small when $\omega_0 = \omega$

Biggest response.

Introduce vector variable

Friday, May 04, 2007
9:05 AM

To solve a 2nd order DEQ $y'' = f(x, y, y')$ (*)
Using Euler or Runga kutta or... first rewrite (*) as 2 1st order eqns

$$y' = v$$

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

$$\dot{x} = v$$

$$v' = y'' = f(x, y, v)$$

$$\dot{v} = \frac{1}{m}(F_0 \cos\omega t - bv - kx)$$

Introduce a vector variable $\vec{u} = \begin{pmatrix} y \\ v \end{pmatrix}$

$$\vec{u}' = \begin{pmatrix} v \\ f(x, \vec{u}) \end{pmatrix} = \vec{F}(x, \vec{u})$$

$$\vec{u} = \begin{pmatrix} x \\ v \end{pmatrix} \quad \dot{\vec{u}} \begin{pmatrix} v \\ \frac{1}{m}(F_0 \cos\omega t - bv - kx) \end{pmatrix}$$

<u>Euler method</u>

$$\vec{u}' = \vec{F}(x, \vec{u})$$

$$\vec{u}' = \lim_{h \to 0} \frac{\vec{u}(x+h) - \vec{u}(x)}{h} = \vec{F}(x, \vec{u})$$

$$\vec{u}(x_{n+h}) = \vec{u}_{n+1} = \vec{u}_n + h\vec{F}(x_n, \vec{u}_n)$$

RK $y' = f(x, y)$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_n, y_n)$$

RK & $\vec{u}' = \vec{F}(x, \vec{u})$

$$\vec{u}_{n+1} = \vec{u}_n + \frac{h}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)$$

$$\vec{k}_1 = \vec{F}(x_n, \vec{u}_n)$$

Forced vibration

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t$$

$$\left.\begin{aligned} r &= -\gamma + i\sqrt{\omega_0^2-\gamma^2} \\ \omega_0 &= \sqrt{\frac{k}{m}} \\ \gamma &= \frac{b}{2m} \end{aligned}\right\} \quad x_h = e^{-\gamma t}\left[c_1\cos\left(\sqrt{\omega_n^2-\gamma^2}\,t\right) + c_2\sin\left(\sqrt{\omega_0^2-\gamma^2}\,t\right)\right]$$

transient pt. of soln $\left\{\begin{array}{l}\text{Homogeneous soln} \to 0 \text{ as} \\ t \to \infty\end{array}\right.$

$$x_p(t) = A\cos\omega t + B\sin\omega t$$

$$A = \frac{F_0(\omega_0^2-\omega^2)}{m\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]}$$

$$B = \frac{2\gamma\omega F_0}{m\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]}$$

$x(t) = x_h(t) + x_p(t)$ as $t \to \infty$ $x(t) \to x_p(t)$

As $t \to \infty$
$x(t) \to x_p(t)$

$$x_p(t) = R\cos(\omega t + \phi)$$

$$\cos(A+B) = \mathrm{Re}\left(e^{i(A-B)}\right) = \mathrm{Re}\left(e^{iA}e^{-iB}\right)$$

"real part"

$$= \mathrm{Re}\left((\cos A + i\sin A)(\cos B - i\sin B)\right)$$

$$= \cos A\cos B + \sin A\sin B$$

$$x_p(t) = R\cos\omega t\cos\phi + R\sin\omega t\sin\phi$$

$$\underbrace{A = R\cos\phi \qquad B = R\sin\phi}$$

$$\tan\phi = \frac{B}{A}$$

$$A^2 + B^2 = R^2$$

$$\phi = \arctan\frac{B}{A}$$

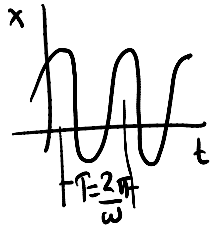
$$R = \sqrt{A^2+B^2}$$

$$R = \left|\sqrt{A^2+B^2}\right|$$

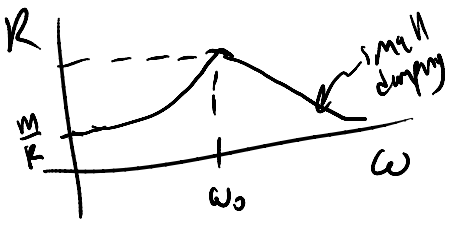
$$R^2 = \frac{F_0^2\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]}{m^2\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^2}$$

$$R = \frac{F_0}{m\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$R \propto F_0$

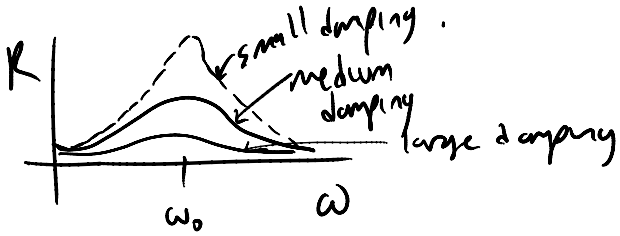


examine the dependence of R on ω



$$R = \frac{F_0}{m} M(\omega;\omega_0,\gamma)$$

$$M = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$



Resonant Response at $\omega=\omega_0$

γ small: large narrow peak in $R(\omega)$
γ larger: smaller, broader peak in $R(\omega)$

Assume $\frac{\gamma}{\omega_0} \ll 1$ & $\omega \approx \omega_0$

$$(\omega_0^2-\omega^2)^2 = (\omega_0-\omega)^2\underbrace{(\omega_0+\omega)^2}_{\approx 2\omega_0} \approx (\omega_0-\omega)^2\, 4\,\omega_0$$

$$M = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} \approx \frac{1}{\sqrt{4\omega_0^2\left[(\omega_0-\omega)^2+\gamma^2\right]}}$$

$(\omega_0-\omega)^2 4\omega_0$ $\qquad$ $\approx 2\omega_0$ $\qquad$ $\approx \omega_0$

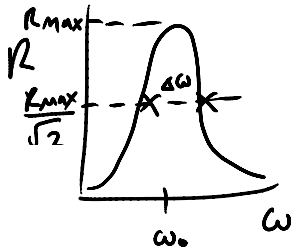
$$\frac{1}{2\omega_0}\,\frac{1}{\sqrt{(\omega_0-\omega)^2+\gamma^2}}$$ ← "Lorentzian" for probability

peak is at $\omega_0 = \omega$

$$R^2_{peak} = \frac{F_0^2}{m^2 4\omega_0^2\gamma^2}$$

Frequency

Monday, May 07, 2007
8:59 AM



$R_{max} \approx R(\omega_0)$ for small γ

$$R = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$$R(\omega_0) = \frac{F_0}{2m\gamma\omega_0} \Rightarrow R_{max}\uparrow \text{ as } \frac{1}{\gamma}$$

Determine $\Delta\omega$ by estimating the values of ω for which $[R(\omega)]^2 \approx \left(\frac{R_{max}}{\sqrt{2}}\right)^2$ $\approx R_{max}$ b/c the peak does not occur exactly at ω_0

$$R^2 = \frac{F_0^2}{m^2((\omega_0^2-\omega^2)^2+4\gamma^2\omega^2)} \qquad \frac{1}{2}R_{max}^2 \approx \frac{F_0^2}{8m^2\gamma^2\omega_0^2}$$

ω_0 by approx

$$(\omega_0^2-\omega^2)^2 = (\omega_0-\omega)^2(\omega_0+\omega)^2 \cong 4\omega_0^2(\omega_0-\omega)^2$$

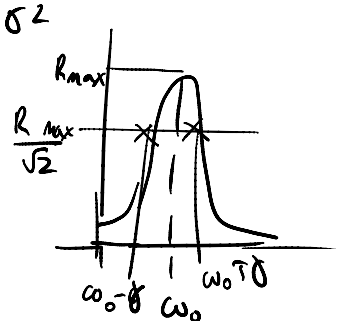
ω_0 by approx

$$\frac{1}{2}R_{max}^2 = R^2 \approx \frac{F_0^2}{m^2 4\omega_0^2[(\omega_0-\omega)^2+\gamma^2]} \approx \frac{F_0^2}{8m^2\gamma^2\omega_0^2}$$

$$(\omega_0-\omega)^2+\gamma^2 \approx 2\gamma^2$$

$$(\omega_0-\omega)^2 = \gamma^2$$

Q: Mon Capitan!



$\Delta\omega \approx 2\gamma$ b/c $\omega_0+\gamma-(\omega_0-\gamma)$

Q = Quality factor

$$= \frac{\omega_0}{\Delta\omega} \approx \frac{\omega_0}{2\gamma}$$

as $\gamma\downarrow$, $Q\uparrow$

Q or $R_{max} \propto \frac{1}{\gamma}$

2nd order linear eqns

Variation of parameters of a 2nd order eqn

$(*)$ $y''+p(x)y'+q(x)y = f(x)$

Assumes we have the gen. soln of $y_h''+p(x)y_h'+q(x)y_h=0$

Assume we have the gen. soln of $y_h'' + p(x)y_h' + q(x)y_h = 0$

$$y_h = c_1 y_1(x) + c_2 y_2(x)$$

Seek a soln of (*) of the form $y_p = v(x)y_1(x) + v_2(x)y_2(x)$

It involves 2 unknown functions
$v_1(x)$ & $v_2(x)$

We'll pick a convenient constraint between v_1 & v_2 to find a particular soln

$$y_p' = \underline{v_1' y_1} + v_1 y_1' + \underline{v_2' y_2} + v_2 y_2'$$

Apply constraint to eliminate underlined terms
$$v_1' y_1 + v_2' y_2 = 0$$

$$y_p' = v_1 y_1' + v_2 y_2'$$
$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plug into (*)

$$y_p'' + p(x)y_p' + q(x)\, y_p = f(x) = v_1 \underbrace{[y_1'' + p(x)y_1' + q(x)y_1]}_{0 \text{ b/c solves homogeneous diff eqn}} + v_2 \underbrace{[y_2'' + p(x)y_2' + q(x)y_2]}_{0} + v_1'[y_1'] + v_2[y_2']$$

$$\left.\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= f(x) \end{aligned}\right\} \text{solve these 2 eqns for } v_1' \text{ \& } v_2'$$

Then integrate

$$y'' + y = \tan(x)$$

1) Solve $y_h'' + y_h = 0$

$$A\cos x + B\sin x = y_h$$
$$v_1(x)\cos x + v_2(x)\sin x = y_p$$
$$v_1'(x)\cos x + v_2'(x)\sin x - v_1(x)\sin x + v_2(x)\cos(x) = y_p'$$

Assume

$$v_1'\cos(x) + v_2'\sin x = 0$$
$$y_p' = -v_1 \sin(x) + v_2 \cos(x)$$
$$y_p'' = -v_1'\sin(x) + v_2'\cos(x) - v_1\cos(x) - v_2\sin(x)$$
$$y_p'' + y_p = v_1[-\cos(x) + \cos(x)]$$

$$+v_2'[\sin(x) - \sin(x)]$$
$$+v_1'[-\sin(x)]$$
$$+v_2'[-\sin(x)] = \tan(x)$$

$$v_1'\cos(x) + v_2'\sin(x) = 0$$
$$v_1'\sin(x) + v_2'\cos(x) = \tan(x)$$

$$\sin(x)[v_1'\cos(x) + v_2'\sin(x)] = 0$$
$$+ \cos(x)[-v_1'\sin(x) + v_2'\cos(x)] = \sin(x)$$

$$v_2' = \sin(x) \Rightarrow v_2 = -\cos(x)$$

$$v_1'\cos(x) + v_2'\cos(x) = 0 \Rightarrow v_1' = \frac{-\sin^2 x}{\cos x}$$

$$v_1 = -\int \frac{\sin^2 x}{\cos x}\,dx$$
$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$
$$= -\int \frac{1}{\cos x}\,dx + \int \cos x\,dx$$
$$= -\ln|\sec(x) + \tan(x)| + \sin(x)$$

$$y = y_h + y_p = [A + v_1]\cos x + [B + v_2]\sin(x)$$
$$= A\cos(x) + B\sin(x) - \ln|\sec(x) + \tan(x)|\cos(x)$$
$$+ \sin(x)\cos(x)$$
$$- \cos(x)\sin(x)$$
$$y = A\cos(x) + B\sin(x) - \ln(|\sec x + \tan x|)\cos x$$

Variation of Parameters, Kirchoff's Laws

Tuesday, May 08, 2007
9:04 AM

Variation of Parameters

let $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$

we know y_1 & y_2 solve eqn

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'$$

Assume $v_1' y_1 + v_2' y_2 = 0$

$$y_p' = v_1 y_1' + v_2 y_2'$$
$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

$$y_p'' + p y_p' + q y_p = v_1 \overbrace{[y_1'' + p y_1' + q y_1]}^{0}$$
$$v_2 [y_2'' + p y_2' + q y_2]$$
$$v_1' [y_1'] \quad 0$$
$$v_2' [y_2'] = f(x)$$

$$v_1' y_1 + v_2' y_2 = 0 \longrightarrow v_1' = -\frac{y_2}{y_1} v_2'$$
$$v_1' y_1' + v_2' y_2' = f(x) \longrightarrow v_2' \left[y_2' - \frac{y_2}{y_1} y_1'\right] =$$

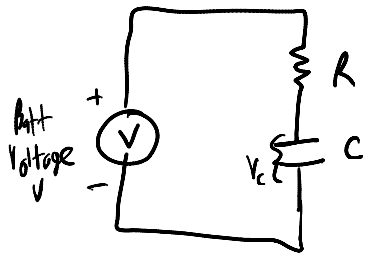
$W(y_1, y_2)$ Wronskian

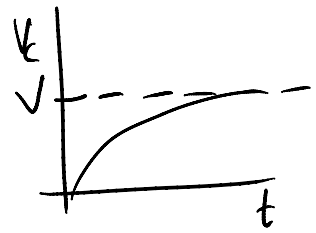
$$v_2' \left[\frac{y_1 y_2' - y_2 y_1'}{y_1}\right] = f(x)$$

$$v_2' = \frac{y_1 f(x)}{W(y_1, y_2)}$$

$$v_1' = -\frac{y_2 f(x)}{W(y_1, y_2)}$$

$$y_p = -y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)} dx$$





Electric Circuit

Simple relationships between V & current I

$$I = \frac{dQ}{dt} \quad (Q = \text{charge})$$

Simple circuit elmts

Resistor (R) $\frac{1}{\Omega}$ ← resistance $\qquad V_R = IR$

large resistance R → small current I for fixed voltage V

Capacitor (C) $\frac{1}{F}$ ← capacitance $\qquad V_C = \frac{Q}{C}$

large capacitance C → large charge Q for given applied voltage V

Inductor (L) $\frac{1}{H}$ ← inductance $\qquad V_L = L\frac{dI}{dt}$

Current generates a $\vec{B}$ field & time varying $\vec{B}$ field generates an E field

Kirchoff's Laws

(1) Voltage drops around any closed circuit must be zero

$$V = V_R + V_C$$
$$(V - V_C - V_R = 0)$$

(2) Current can't be created or destroyed so total current going into a node or out of a node must be the same

Parameters are: Applied Voltage V, Resistance R, & C

We want an eqn for V_c

$$V = V_R + V_c$$
$$V_R = IR$$
$$V_c = \frac{Q}{C}$$

we want to eliminate V_R from the eqn

$$V = IR + V_c$$

$$\frac{dQ}{dt} = \frac{d}{dt}(CV_c) = C\frac{dV_c}{dt}$$

$$V = RC\frac{dV_c}{dt} + V_c$$

$$\frac{dV_c}{dt} = \frac{1}{RC}(V - V_c) \quad ; \quad V_c(0) = 0$$

$$\int \frac{1}{V - V_c}\, dV_c = \int \frac{1}{RC}\, dt$$

$$-\ln|V - V_c| = \frac{t}{RC} + A$$

$$|V - V_c| = e^{-\frac{t}{RC} - A} \Rightarrow V - V_c = K e^{-\frac{t}{RC}}$$

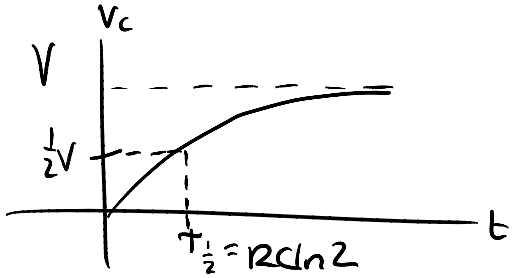
$$V_c(t) = V - Ke^{-\frac{t}{RC}}$$

$$V_c(0) = V - Ke^{-\frac{0}{RC}} = 0$$

$$V - K = 0$$

$$V = K$$

$$V_c = V(1 - e^{-\frac{t}{RC}})$$



$$V_c(\tfrac{1}{2}) = \tfrac{1}{2}V = V(1 - e^{-\frac{T_{\frac{1}{2}}}{RC}})$$

$$\frac{1}{2} = 1 - e^{-\frac{T_{\frac{1}{2}}}{RC}}$$

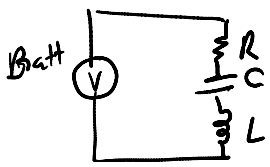
$$\frac{1}{2} = e^{-\frac{T_{\frac{1}{2}}}{RC}}$$

$$\frac{-T_{\frac{1}{2}}}{RC} = \ln\frac{1}{2} = -\ln 2$$

LRC Circuit

Wednesday, May 09, 2007
8:58 AM

LRC circuit



want an eqn for V_c = voltage drop across capacitor in terms of the parameters V, R, C, L

$$V = V_R + V_C + V_L$$

$V_R = I\,R$ not I_R b/c I is =.

$$V_C = \frac{Q}{C}$$

$$V_L = L\frac{dI}{dt}$$

$$\frac{dV_C}{dt} = \frac{d}{dt}\left(\frac{Q}{C}\right) = \frac{1}{C}\frac{dQ}{dt} = \frac{I}{C} = \frac{1}{RC}V_R = \frac{1}{RC}(V - V_C - V_L)$$

$$= \frac{1}{RC}\left(V - V_C - L\frac{dI}{dt}\right)$$

$$V_C = \frac{Q}{C}$$

$$\frac{dV_C}{dt} = \frac{dQ}{dt}\frac{1}{C} = \frac{1}{C}I$$

$$C\frac{d^2V_C}{dt^2} = \frac{dI}{dt}$$

$$\frac{dV_C}{dt} = \frac{1}{RC}\left(V - V_C - LC\frac{d^2V_C}{dt^2}\right)$$

$$= \frac{1}{RC}V - \frac{1}{RC}V_C - \frac{1}{R}\frac{d^2V_C}{dt^2}$$

$$\boxed{\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_C}{dt^2} + \frac{1}{LC}V_C = \frac{1}{LC}V}$$

particular soln: $V_C = V$

homogeneous soln $V_C = V_h$ $\qquad V_C = V + V_h$

$$\frac{d^2V_h}{dt^2} + \frac{R}{L}\frac{dV_h}{dt} + \frac{1}{LC}V_h = 0$$

$$r^2 + \frac{R}{L}r + \frac{1}{RC} = 0$$

$$r_\pm = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

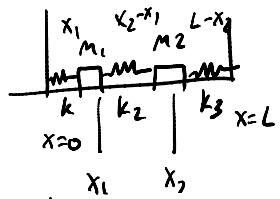
$\frac{1}{LC} > \frac{R^2}{4L^2} \Rightarrow$ decaying oscillations

$\frac{1}{LC} = \frac{R^2}{4L^2} \Rightarrow$ critically damped

$\frac{1}{LC} < \frac{R^2}{4L^2} \Rightarrow$ overdamped

Systems of 1st order lin eqns

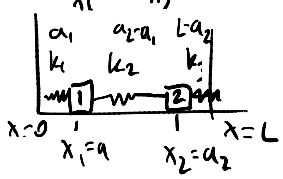
Ex



$$m_1 \frac{d^2x_1}{dt^2} = F_1 = -k_1(x_1-a_1) + k_2((x_2-x_1)-(a_2-a_1))$$

$$m_2 \frac{d^2x_2}{dt^2} = F_2 = -k_2((x_2-x_1)-(a_2-a_1)) + k_3((L-x_2)-(L-a_1))$$

equilib



let $x_1 = a_1 + y_1 \Rightarrow y_1 = x_1 - a_1$ $\qquad \frac{d^2x_1}{dt^2} = \frac{d^2y_1}{dt^2}$

$x_2 = a_2 + y_2 \Rightarrow y_2 = x_2 - a_1$ $\qquad \frac{d^2x_2}{dt^2} = \frac{d^2y_2}{dt^2}$

$$m_1 \frac{d^2y_1}{dt^2} = -k_1 y_1 + k_2(y_1 - y_2)$$

$$m_2 \frac{d^2y_2}{dt^2} = k_2(y_1 - y_2) - k_3 y_2$$

$$\boxed{\begin{aligned} m_1 \frac{d^2y_1}{dt^2} &= -(k_1+k_2)y_1 + k_2 y_2 \\ m_2 \frac{d^2y_2}{dt^2} &= -(k_2+k_3)y_2 + k_2 y_1 \end{aligned}}$$

$$y_1 = \frac{m_2}{k_2}\frac{d^2y_2}{dt^2} + \frac{(k_2+k_3)}{k_2} y_2$$

$$m_1 \frac{d^2}{dt^2}\left(\frac{m_2}{k_2}\frac{d^2y_2}{dt^2} + \frac{(k_2+k_3)}{k_2} y_2\right) = -(k_1+k_2)\left(\frac{m_2}{k_2}\frac{d^2y_2}{dt^2} + \frac{k_2+k_3}{k_2} y_2\right) + k_2 y_2$$

Variables are:

$y_1, v_1 = \frac{dy_1}{dt}, \quad y_2, v_2 = \frac{dy_2}{dt}$

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dv_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} v_1 \\ -\frac{(k_1+k_2)}{m_1} y_1 + \frac{k_2}{m_1} y_2 \\ v_2 \\ -\frac{(k_2+k_3)}{m_2} y_2 + \frac{k_2}{m_2} y_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{(k_2+k_3)}{m_2} & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{bmatrix}}_{\vec{U}}$$

$$\frac{d\vec{U}}{dt} = M\vec{U}$$

Seek solns in the form $\vec{U} = \vec{U}_0 e^{\lambda t}$; $\vec{U}' = \lambda U_0 e^{\lambda t}$

const vector

$\lambda \vec{U}_0 e^{\lambda t} = M \vec{U}_0 e^{\lambda t}$

$\lambda \vec{U}_0 = M \vec{U}_0$ $\qquad$ λ is Eigenvalue of Matrix

Matls

Friday, May 11, 2007
8:00 AM

Can a material fail under pure compression? Yes.

Coordination # & Ionic radii.

As $\frac{r_{cation}}{r_{anion}}$ # gets bigger, coordination # gets bigger

Site Selection 2

Stoichiometry

Charge Neutrality (Net charge in structure should be zero)

If all of 1 type of sites is full the remainder have to go somewhere else.

Ex. FCC unit cell has $4 O_H$ & $8 T_D$ sites.

Suppose for a specific ceramic: i) anions form FCC sublattice, ii) each unit cell has 6 cations & iii) 2 cations prefer O_H sites.

What is fraction of O_H & T_D sites filled?

$$\frac{4 \text{ anions in FCC}}{1 + 12\left(\frac{1}{4}\right) = 4 \text{ octahedral sites}}$$

$8\ T_D$ sites

All O_H site full, 100%

2 cations in T_D sites. $\frac{2}{8} = 25\%$

What would be stoichiometry for above ceramic? What are possible values of ionic charges on each cation/anion?

4 anions / 6 cations

$M_6 A_4$ or $Metal_3^{2+} Anion_2^{3-}$

Site Selection 3

Bond Hybridization.

Significant covalent bonding

Significant covalent bonding

- hybrid orbitals can have significant impact if significant covalent bond character present.

Predicting the structure of FeO

look up r cation & anion

Calculate $\frac{r_c}{r_a}$

look up $\frac{r_c}{r_a}$ in relation to coordination # & structure

Rock Salt Structures: Same concepts can be applied to ionic solids in general.

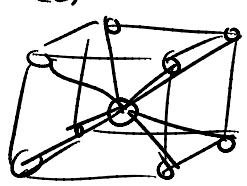
MgO + FeO also have NaCl structure.

cations prefer O_h sites

AX Crystal structures

zinc blende, CsCl

Cesium Chloride



$\frac{r_{Cs^+}}{r_{Cl^-}} = .939$ ∴ cubic sites preferred

Zinc Blende

uber-complicated cube

$\frac{r_{Zn^{2+}}}{r_{O^{2-}}} = .529 = O_H$??

No! It's not!

Predicted O_H, but observed T_D.

Due to bonding hybridization, zinc prefers T_D

Fluorite structure

another uber-complicated diagram

Calcium Fluoride (CaF_2)

Cations in cubic sites

UO_2, ThO_2, ZrO_2, CeO_2

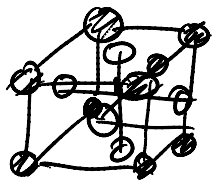
anti-fluorite structure - cations & anions reversed

with Fluorite structure ...
anions reversed

Simple cubic

If you reverse Fluorite, it doesn't work. Works for CsCl, but not for Fluorite, called anti-fluorite.

Perovskite
$BaTiO_3$



$Ti^{4+} \rightarrow \#6$
$Ba^{2+} \rightarrow \#12$
$O^{2-} \rightarrow \#2$

Ceramics are more brittle than metals b/c they have slip systems. Too much energy is needed to move one anion past another in ionic, and so slippage is difficult.

Ceramics $\rho = \dfrac{n'(\Sigma A_C + \Sigma A_A)}{V_C N_A}$ always compounds so need to calculate $A_C + A_A$

Metals $\rho = \dfrac{n' A}{V_C N_A}$

Silicate Ceramics
Most common elmts on Earth are Si & O

SiO_2 structures are quartz, crystabolite, & tridymite
Strong Si-O bond leads to a strong high melting material

We can also form amorphous structures.

A lot of glass is Si-O. - Silica Gels SiO_2, Si^{4+} & O^{2-} not in well-ordered lattice
- Charge balanced by H^+ to form OH^- at dangling bonds.
high surface area $>200 m^2/g$

Carbon Forms

Carbon black-amorphous - surface area ca. 1000 m^2/g

Diamond
- tetrahedral carbon
  - hard - no good slip planes
  - brittle - can cut it
- large diamonds - jewelry
- small diamonds
  - often man-made, used for cutting tools & polishing
- diamond films - tools, medical devices, etc. (coatings)

Graphite
- layer structure - aromatic layers
- weak van der Waals forces between layers
- planes slide easily, good lubricant.
- stable form of Diamond

Fullerenes & Nanotubes
- wrap graphite sheet by curving into a ball or tube
- Buckminster fullerenes (Bucky Ball)
  Like a soccer ball C_{60} - also C_{70} + others

Defects in Ceramic Structures

Substitutional Impurity
- will substitute for electronically most similar state. Cation for cation site

Interstitial impurity
- impurity must be relatively small compared to anion)
  anion interstitials just rare.

Frenkel Defect pairs
- cation out of place. Cation vacancy + cation interstitial

Shottky Defect pairs
- paired set of cation & anion vacancies.

Equilib conc. of defect pairs $\sim e^{-q_d/2kT}$

<u>Impurities</u>

Impurities must satisfy charge balance - electroneutrality

NaCl Na^+ Cl^-

Take out 2 Na's to satisfy
Ca^{2+} w/ Ca^{2+} impurity

Resulting geometry is a cation vacancy.

Ceramic Phase diagrams

Measuring elastic Modulus

Linear Algebra reviews

$$-2x_1 + 3x_2 - x_3 = 4$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$x_1 - 5x_2 + 3x_3 = -1$$

$$\underbrace{\begin{bmatrix} -2 & 3 & -1 \\ -1 & 1 & 2 \\ 1 & -5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}}_{\vec{b}}$$

You can only find a soln if the determinant $\neq 0$

If $A\vec{x} = \vec{b}$ has a unique soln
$\vec{x} = A^{-1}\vec{b}$ if determinant of $A \neq 0$

Consider a homogeneous problem

$$\underset{\approx}{A}\,\vec{x} = \vec{0}$$

$\underset{\approx}{A} = n \times n$ matrix

this has a unique soln if $\det(A) \neq 0$ which is $\vec{x} = \vec{0}$

There are nonzero (nontrivial) solns to $A\vec{x} = \vec{0}$ only if $\det(A) = 0$

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix}$$

$$\det(A) = 1(-11-3) - 2(-22+1) - 4(6+1)$$
$$= -14 + 42 - 28$$
$$= 0$$

Since $\det(A) = 0$, then there are nonzero solns.

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -4 \\ 0 & -3 & 9 \\ 0 & 5 & -15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$ upper triangular form but bottom is all zeros.

2nd row -2(first row)
Third row + 1st row

$$x_1 + 2x_2 - 4x_3 = 0$$
$$x_2 - 3x_3 = 0$$

x_3 arbitrary
$x_2 = 3x_3$
$x_1 = -2x_2 + 4x_3$
$-2(3x_3) + 4x_3 = -2x_3$

Soln to $A\vec{x} = \vec{0}$
is $\vec{x} = C \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

Linear Algebra Review

Def. of lin. ind. of vectors $\rightarrow$ A set of k vectors $\vec{x}_1, \vec{x}_2, \vec{x}_k$ is lin. <u>dependent</u> if there exist constants $c_1, c_2 \ldots c_k$ not all 0 s.t. $c_1x_1 + c_2x_2 + \cdots c_kx_k = 0$.
if the only soln to $c_1x_1 + c_2x_2 \ldots c_kx_k = 0$ is $c_1 = c_2 = c_k = 0$ then the vectors are lin. independent.

let $\vec{x}_1 \ldots \vec{x}_k$ be n-dimensional vectors

if $k < n$, $\Rightarrow$ could either be lin. ind. or dep.

if $k > n$, $\Rightarrow$ must be lin. dep.

Focus on $k = n$

n vectors, n dimensional, $k=n$

Rewrite $c_1\vec{x}_1 + c_2\vec{x}_2 + c_n\vec{x}_n = \vec{0}$ in matrix form as follows

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix}}_{\underset{\sim}{X}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underset{\sim}{X}\vec{c} = \vec{0}$$

if $\det(\underset{\sim}{X}) \neq 0 \Rightarrow$ unique soln $\vec{c}=0$

if $\det(\underset{\sim}{X}) = 0 \Rightarrow$ nontrivial soln for $\vec{c}$ exists

if $\det(\underset{\sim}{X}) \neq 0 \Rightarrow$ vectors $\vec{x}_1 \ldots \vec{x}_k$ are lin ind.

if $\det(\underset{\sim}{X}) = 0 \Rightarrow$ vectors are lin dep.

$\det(\underset{\sim}{X}) = 0 \Rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$ are lin dep

we solved $\underset{\sim}{X}\vec{c} = \vec{0} \Rightarrow \vec{c} = C\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

there exist constants $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$

$-2 \quad 3 \quad 1$

$$-2\vec{x}_1 + 3\vec{x}_2 + \vec{x}_3 = 0$$

Linear systems of 1st order homogeneous deqs

$$* \quad \frac{d\vec{x}}{dt} = \underset{\approx}{P(t)}\,\vec{x}$$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, P is an $n \times n$ matrix

This has a superposition principle:

if $\vec{x}_1(t)$ & $\vec{x}_2(t)$ solve the eqn * then so does $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ for any $c_1\ c_2$

Check $\frac{d\vec{x}_1}{dt} = \underset{\approx}{P}(t)\vec{x}_1$ & $\frac{d\vec{x}_2}{dt} = \underset{\approx}{P}(t)\vec{x}_2$

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1\frac{d\vec{x}_1}{dt} + c_2\frac{d\vec{x}_2}{dt}$$

$$= c_1\underset{\approx}{P}(t)\vec{x}_1 + c_2\underset{\approx}{P}(t)\vec{x}_2$$

$$= \underset{\approx}{P}(t)(c_1\vec{x}_1 + c_2\vec{x}_2)$$

QED

Eigenvalues

Monday, May 14, 2007
8:59 AM

Let $\underset{\sim}{A}$ be an $n \times n$ matrix. λ is an eigenvalue of $\underset{\sim}{A}$ and $\vec{v}$ is its associated eigenvector if $\underset{\sim}{A}\vec{v} = \lambda\vec{v}$, $\vec{v} \neq 0$,
Here λ is a scalar + $\vec{v}$ is an n dimensional vector

$$(*) \quad \frac{d\vec{x}}{dt} = \underset{\sim}{P}(t)\,\vec{x}$$

$\vec{x}$ is an n-dim. vector
$\underset{\sim}{P}$ is an $n \times n$ matrix

The gen. soln of $*$ involves n lin. ind. solns
$\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)$

$$\vec{x}(t) = c_1\vec{v}_1(t) + c_2\vec{v}_2(t) + \ldots + c_n\vec{v}_n(t)$$

Want a simple test for whether $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are lin. ind.
If we have the gen soln then we know we can solve initial value prob $*$ + $\vec{x}(t_0) = \vec{x}_0$

$$\vec{x}(t_0) = c_1\vec{v}_1(t_0) + c_2\vec{v}_2(t_0) + \ldots + c_n\vec{v}_n(t_0) = \vec{x}_0$$

Rewrite in matrix form

$$c_1\vec{v}_1 + \cdots c_n\vec{v}_n = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1(t_0) & \vec{v}_2(t_0) & \cdots & \vec{v}_n(t_0) \\ | & | & & | \end{bmatrix}}_{\underset{\sim}{V}} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{\vec{c}} = \underbrace{\begin{bmatrix} x_{0,1} \\ \vdots \\ x_{0,n} \end{bmatrix}}_{\vec{x}_0}$$

Solve $\underset{\sim}{V}\vec{c} = \vec{x}_0$ for $\vec{c}$

We can always do this $(\vec{c} = \underset{\sim}{V}^{-1}\vec{x}_0)$ if $\det(\underset{\sim}{V}) \neq 0$

Wronskian $W(t) = \mathrm{Det}\begin{bmatrix} | & | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \\ | & | & | & & | \end{bmatrix}$

if $W(t) \neq 0 \Rightarrow$ solns are lin. ind.
if $W(t) = 0 \Rightarrow$ solns are lin. dep.

Can show that the W is always zero or never zero, hence it's enough to check @ 1 pt

$$y''' + ay + by' + cy = 0$$

$$W = \mathrm{Det}\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$$

$y_1 = x_1$
$y_1' = x_2$
$y_1'' = x_3$

$x_1' = x_2$
$x_2' = x_3$
$\underbrace{x_3' = -a x_3 - b x_2 - c x_1}_{\text{Wronskian will be}}$

$$W = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

<u>ex</u>

$\frac{dx_1}{dt} = 4x_1 - 5x_2$

$\frac{dx_2}{dt} = 2x_1 - 3x_2$

$x_1(0) = 8 \quad x_2(0) = 5$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2' \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \underset{\approx}{A}\vec{x}; \quad \vec{x}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$\underset{\approx}{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

If instead we had just $\frac{dx}{dt} = ax$,
$x(0) = x_0$

$$x = x_0 e^{at}$$

But anyway,

$$\vec{x}(t) = e^{\underset{\approx}{A}t}\vec{x}_0$$

this is a matrix And we don't really know what that is.

Instead, let's look for a soln $\vec{x} = e^{\lambda t}\vec{v}$ $\quad \frac{d\vec{x}}{dt} = \underset{\sim}{A}\vec{x}$

λ = scalar

$\vec{v}$ = const. vector

$$\frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v}$$

$$\frac{d\vec{x}}{dt} = \underset{\sim}{A}\vec{x}$$

$$\vec{v} = \underset{\sim}{A} e^{\lambda t}\vec{v}$$

$$\boxed{\lambda\vec{v} = A\vec{v}}$$ ← need to solve this eigenvalue prob.

First find eigenvalues ⇒ find eigenvectors

Rewrite this eqn

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\vec{v} = \vec{0}$$

nxn identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{I}\vec{v} = \vec{v}$$

Eigenvalue/eigenvector problem $\underset{\sim}{A}\vec{v} = \lambda\vec{v}$

Find eigenvalues:

$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\vec{v} = \vec{0}$ has a non-zero soln only if

$\mathrm{Det}(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$ ← this is an eqn for λ

<u>ex</u>

$$\frac{d\vec{x}}{dt} = \underset{\sim}{A}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$\underset{\sim}{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

Find eigenvalues of $\underset{\sim}{A}$

$$\mathrm{Det}\left(\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \mathrm{Det}\begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$$

$$= (4-\lambda)(-3-\lambda) + 10 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

Characteristic polynomial

$$(\lambda - 2)(\lambda + 1) = 0$$

2 eigenvalues ; $\lambda_1 = 2, \lambda_2 = -1$

$\lambda_1 = 2$: $\quad \underset{\sim}{A}\,\vec{v}_1 = 2\vec{v}_1 \quad \Rightarrow (\underset{\sim}{A} - 2\underset{\sim}{I})\,\vec{v}_1 = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\lambda_1 = -1$: $\quad \underset{\sim}{A}\,\vec{v}_2 = -\vec{v}_2 \qquad\qquad = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$2y_1 - 5y_2 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Find eigenvectors by solving
$(\underset{\sim}{A} - \lambda \underset{\sim}{I})\,\vec{v} = \vec{0}$

$$(\underset{\sim}{A} - \lambda_2 \underset{\sim}{I})\,\vec{v}_2 = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvectors

Tuesday, May 15, 2007
8:56 AM

$\underset{\approx}{A}$ = n×n vector
λ = scalar (eigenvalue of $\underset{\approx}{A}$)
$\vec{v}$ = n dimensional vector (eigenvector)

$$\underset{\approx}{A}\vec{v} = \lambda\vec{v}, \quad \vec{v} \neq \vec{0}$$

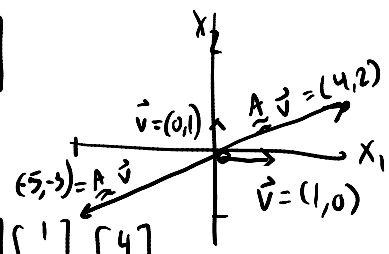
1) $(\underset{\approx}{A} - \lambda I)\vec{v} = \vec{0}$

2) $\underset{\approx}{A} - \lambda I$ is "a singular matrix". Cannot be inverted

3) $\mathrm{Det}(A - \lambda I) = 0 \Leftarrow$ eqn for eigenvalue λ

Geometric interpretation of eigenvectors ($\underset{\approx}{A}\vec{v} = \lambda$; $A\vec{v}$ & $\vec{v}$ are either parallel or antiparallel provided $\lambda \neq 0$ & λ is real)

$$\underset{\approx}{A} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$



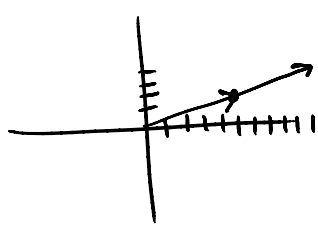
$$\underset{\approx}{A}\vec{v} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

$$\underset{\approx}{A}\vec{v} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

These are not eigenvectors b/c the vectors don't pt. in same direction as multiplier

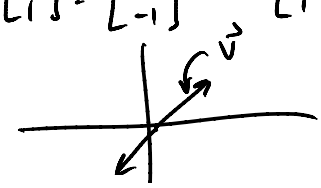
$$\underset{\approx}{A}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 20-10 \\ 10-6 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

eigenvalue eigenvector of A

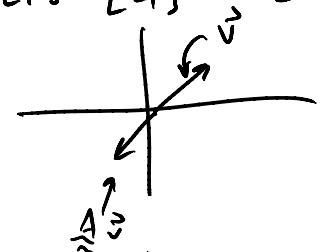


When you multiply by the matrix you get a vector pointing in some direction

$$\underset{\approx}{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$\underset{\approx}{A}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4 & -5\\2 & -3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix} = -\begin{bmatrix}1\\1\end{bmatrix}$$



Eigenvalues of the identity matrix $= 1$

$$\frac{d\vec{x}}{dt} = \underset{\approx}{A}\vec{x} \quad ; \underset{\approx}{A} = \begin{bmatrix}4 & -5\\2 & -3\end{bmatrix}$$

$$\vec{x} = c_1 e^{2t}\begin{bmatrix}5\\2\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

λ $\quad$ $\lambda = -1$

Check: $\frac{d\vec{x}}{dt} = 2c_1 e^{2t}\begin{bmatrix}5\\2\end{bmatrix} - c_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

$$\underset{\approx}{A}\vec{x} = c_1 e^{2t}\underbrace{\underset{\approx}{A}\begin{bmatrix}5\\2\end{bmatrix}}_{2\begin{bmatrix}5\\2\end{bmatrix}} + c_2 e^{-t}\underbrace{\underset{\approx}{A}\begin{bmatrix}1\\1\end{bmatrix}}_{-1\begin{bmatrix}1\\1\end{bmatrix}} \quad \checkmark$$

<u>Defs</u> $\quad \underset{\approx}{A} = n\times n$ matrix

$$\mathrm{Det}(\underset{\approx}{A} - \lambda \underset{\approx}{I}) = c(\lambda)$$

$c(\lambda)$ is an nth order polynomial in λ called "characteristic polynomial"

$c(\lambda) = 0$ "Characteristic eqn"

<u>ex</u> $\underset{\approx}{A} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

eigenvalues & eigenvectors

eigenvalues λ $\quad \mathrm{Det}(\underset{\approx}{A} - \lambda \underset{\approx}{I}) = \mathrm{Det}\begin{bmatrix}-\lambda & 1\\1 & -\lambda\end{bmatrix} = \lambda^2 - 1 = 0$

$\lambda_1 = 1$

$\lambda_2 = -1$

$$(\underset{\approx}{A} - \lambda \underset{\approx}{I})\vec{v} = \vec{0}$$

$\lambda_1 = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$

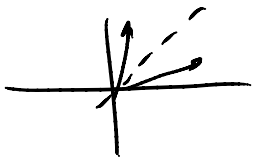
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underset{\approx}{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\vec{x} = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underset{\approx}{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

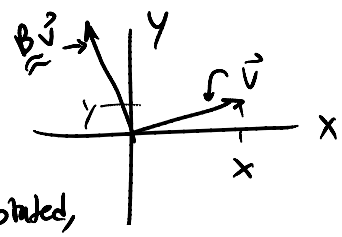


reflects vectors across $y = x$

<u>ex</u> $\underset{\approx}{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ eigenvalues $\operatorname{Det}\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$

$$\lambda_1 = i, \quad \lambda_2 = -i$$

$$\underset{\approx}{B}\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$



rotates vectors by $\frac{\pi}{2}$ CCW

No real eigenvectors

every pt is rotated, $\perp$ to original.

Complex eigenvalues don't lead to real eigenvectors $\rightarrow$ complex eigenvectors

$$e^{i\frac{\pi}{2}}$$

entries in $\underset{\approx}{A}$ are real

characteristic polynomial $c(\lambda)$ has real coefficients

if $\lambda = a+ib$ is a soln of $c(\lambda)=0$ (a, b, real #s)
then so is $\lambda^* = a-ib$. λ^* is complex conjugate of λ

if $\vec{v}$ = eigenvector of $\underset{\sim}{A}$ associated w/ complex eigenvalue $\lambda = a+ib$
$\Rightarrow$ $\vec{v}$ is complex

$\vec{v}^*$ = complex conjugate of $\vec{v}$ is the eigenvector for λ^*

Ex problems, Eigenspaces

Wednesday, May 16, 2007
9:02 AM

ex complex roots to characteristic eqn

$$\underset{\approx}{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\mathrm{Det}(\underset{\approx}{A} - \lambda \underset{\approx}{I}) = \mathrm{Det}\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$$

$$[\underset{\approx}{A} - \lambda \underset{\approx}{I}]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = i: \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -ix - y \\ x - iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = i;\ \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \lambda_2 = -i;\ \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i \quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ix - y \\ x + iy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For complex eigenvalues $\lambda_1 = \alpha + ib$ & $\lambda_2 = \alpha - ib$, eigenvectors can be chosen so that $\vec{v}_2 = \vec{v}_1^*$

$$\frac{d\vec{x}}{dt} = \underset{\approx}{A}\vec{x}$$

$$\vec{x} = c e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix} + c^* e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

c = complex # = $a + ib$

c^* = complex conjugate

($\bar{c}$ is another notation for c^*)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (a + ib)(\cos t + i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix} + (a - ib)(\cos t - i\sin t)\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} ia\cos t - a\sin t - b\cos t - ib\sin t \\ a\cos t + ia\sin t + ib\cos t - b\sin t \end{bmatrix} + \begin{bmatrix} -ia\cos t - a\sin t - b\cos t + ib\sin t \\ a\cos t - ia\sin t - ib\cos t - b\sin t \end{bmatrix}$$

$$= \begin{bmatrix} -2a\sin t - 2b\cos t \\ 2a\cos t - 2b\sin t \end{bmatrix}$$

$\lfloor 2a\cos t \;\; -2b\sin t \rfloor$

$$\vec{x} = A\begin{bmatrix}\sin t\\ -\cos t\end{bmatrix} + B\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} \quad \begin{pmatrix}-2a = A\\ -2b = B\end{pmatrix}$$

Simple test for whether the vectors are real or not:

Check that $\vec{x} = \vec{x}^*$

this only happens if there's no imaginary part.

$$\vec{x} = ce^{it}\vec{v}_1 + c^* e^{-it}\vec{v}_1^*$$

$$\vec{x}^* = c^* e^{-it}\vec{v}_1^* + ce^{it}\vec{v}_1 \quad \checkmark$$

We're solving

$$\frac{d\vec{x}}{dt} = \underset{\approx}{A}\vec{x}$$

$$\vec{x}(0) = \vec{x}_0$$

Soln: $\vec{x}(t) = n$-dim vector

$\underset{\approx}{A} = n \times n$ matrix

<u>Case 1</u>

$\underset{\approx}{A}$ has n <u>distinct</u> <u>real</u> eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$ with associated eigenvectors $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n$

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \ldots + c_n e^{\lambda_n t}\vec{v}_n$$

where we choose $c_1, \ldots c_n$ so that $\vec{x}_0 = c_1\vec{v}_1 + \ldots c_n\vec{v}_n$

<u>Case 2</u>

$\underset{\approx}{A}$ has n <u>distinct</u> <u>roots</u>: for each complex conjugate pair $\lambda = a + ib$, $\lambda^* = a - ib$ there is an associated soln

$$ce^{at}e^{ibt}\vec{v}_1 + c^* e^{at}e^{-ibt}\vec{v}_1^*$$

eigenvector $\vec{v}_1$ eigenvector $\vec{v}_1^*$

repeated roots of characteristic eqn

<u>ex 1</u>

$$\underset{\approx}{A} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \det\begin{bmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{bmatrix} = (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

$\lambda_2 = 1$

repeated eigenvalue

$$\begin{bmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

any vector is an eigenvector

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ is also an eigenvector

$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$ $\lambda_2 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

choose these b/c they are lin ind.

<u>ex2</u>

$\underset{\approx}{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ lower triangular matrix

"defective"

$\det(\underset{\approx}{B} - \lambda \underset{\approx}{I}) = \det\begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0$

$\lambda = 1$ is a repeated eigenvalue

$\underset{\approx}{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\lambda_1 = 1$ $\lambda_2 = 1$

$(\underset{\approx}{B} - \lambda \underset{\approx}{I})\vec{v} = \vec{0}$; $\lambda = 1$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0$

x must $= 0$ $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

There is only one eigenvector in this problem

$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ only one.

<u>Eigenspaces</u>

1. The set of al eigenvectors associated w/ an eigenvalue λ is a subspace of our n-dimensional space called the <u>eigenspace</u> of λ.

Subspaces are lines, planes, origin, anything where you take 2 vectors & their lin. combination is also in that subspace.

Set of all eigenvectors forms a subspace.
Lin combo of 2 eigenvectors w/ same eigenvalue, you get an eigenvector in same eigenspace.

The # of times λ is repeated as a root of $c(\lambda) = 0$ is called the algebraic multiplicity of λ

called the <u>algebraic multiplicity</u> of λ.

The dimension of the eigenspace of λ (# of lin. ind. eigenvectors) is called the geometric multiplicity of λ.

Phase plane analysis

Monday, May 21, 2007
8:57 AM

Phase plane analysis (read Sections 6.1–6.2)

$$\left.\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}\right\} (*)$$

2 1st-order autonomous eqns. Don't depend on ind. variable.
F & G do not depend on t

<u>Plan</u>: (Qualitative) Plotting of soln curves $(x(t), y(t))$ in the (x,y) plane

(x,y) plane = Phase plane
Soln curves can be parameterized by t:
$(x,y) = (x(t), y(t))$

The vector field $(F(x,y), G(x,y))$ defines the tangent lines to the soln curves.

Simplest type of curve is a pt. (x^*, y^*) for which $F(x^*,y^*) = G(x^*,y^*) = 0$ Critical pts: (x^*, y^*) of (F,G)

Corresponds to an equilib soln of $*$ since
$$\frac{dx^*}{dt} = \frac{dy^*}{dt} = 0$$

<u>Simple examples</u>

$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = ky$$
$k \neq 0$

$(0,0)$ is a critical pt. It is the only critical pt. If $k=0$, then critical pts are $x=0$, $y=$ anything.

$$x = x_0 e^t$$
$$y = y_0 e^{kt}$$

Method 1 for getting $y = y(x; x_0, y_0)$ eliminate t
Solve $x = x_0 e^t$ for t
$e^t = \frac{x}{x_0}$ provided $x_0 \neq 0$

$$y = y_0 (e^t)^k = \frac{y_0}{x_0^{\,k}} x^k$$

Method 2

$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = ky$$

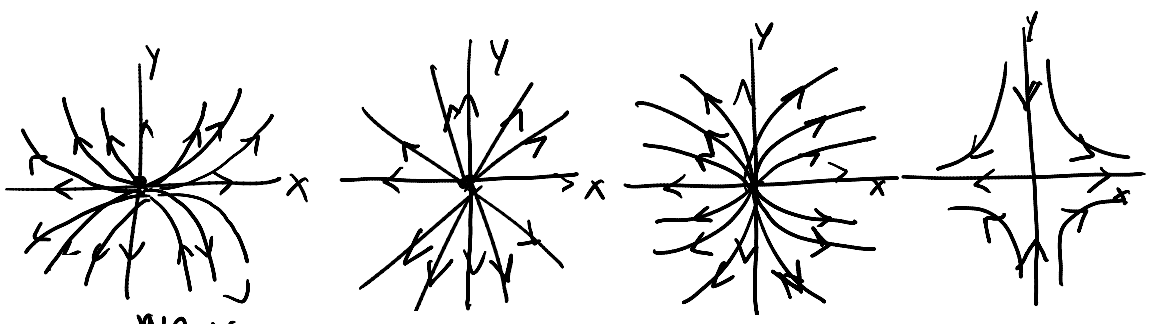
$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{ky}{x}$$

$$\int \frac{dy}{y} = \int k \frac{dx}{x}$$

$$\ln y = k \ln x + C$$

$$y = e^{k \ln x} e^C = A x^k$$

$k > 1$	$k = 1$	$0 < k < 1$	$k < 0$



origin is unstable critical pt "unstable node"

origin is an "unstable" node

unstable node

still unstable node. Saddle pt b/c it is stable in one pt

$$m \frac{d^2x}{dt^2} = -kx$$

"phase portrait"

A critical pt (x^*, y^*) of the autonomous system $(*)$ is <u>stable</u> provided that if the initial condition (x_0, y_0) is sufficiently close (x^*, y^*) then $(x(t), y(t))$ remains close to the critical pt (x^*, y^*) for all $t > 0$.

Formally, let $\vec{x}(t) = (x(t), y(t))$, $\vec{x}_0 = (x_0, y_0)$, $\vec{x}^* = (x^*, y^*)$

$\vec{x}^*$ is stable provided that for each $\varepsilon > 0$, there exists a $\delta > 0$ s.t. if $|\vec{x}_0 - x^*| < \delta$, then $|\vec{x}(t) - \vec{x}^*| < \varepsilon$

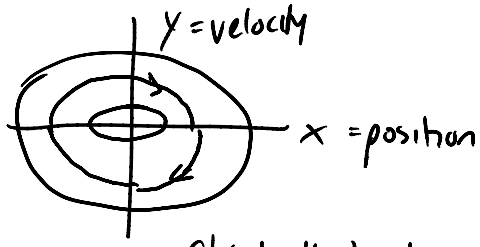
$\forall t > 0$

If a C.P. is not stable $\rightarrow$ it is <u>unstable</u>.
If a C.P. is stable and $\vec{x}(t) \rightarrow \vec{x}^*$ as $t \rightarrow \infty$ (provided $\vec{x}_0$ close enough to $\vec{x}^*$) then $\vec{x}^*$ is asymptotically stable.

$$m \frac{d^2x}{dt^2} = -kx$$

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\frac{k}{m} x$$



closed, elliptical curves

Phase portrait for linear spring

Tuesday, May 22, 2007
9:01 AM

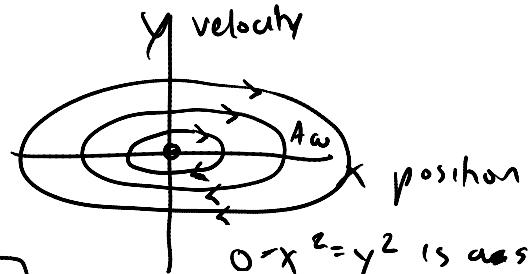
Tues 11-1 Fri 1-3 (not on Wed)

Phase portrait for linear spring

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{dx}{dt} = y = \text{velocity of spring}$$

$$\frac{dy}{dt} = -\frac{k}{m}x = \text{acceleration of spring}$$



When velocity is neg, x is dec. So it has to be CW, not CCW.

$\boxed{\omega < 1}$

$0 - x^2 = y^2$ is a stable C.P. but not asymptotically stable.

We can obtain the phase portrait by considering the eqn for energy conservation: (no damping)

$$\text{total } E = KE + PE = \frac{1}{2}my^2 + \frac{1}{2}kx^2$$

$$\frac{1}{2}my_0^2 + \frac{1}{2}kx_0^2 = E_0 = \frac{1}{2}my^2 + \frac{1}{2}kx^2$$

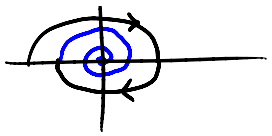
$$x(t) = A\cos(\omega t - \alpha) \quad ; \quad \omega = \sqrt{\frac{k}{m}}$$

$$y(t) = \frac{dx}{dt} = -A\omega\sin(\omega t - \alpha)$$

$$x^2 + \frac{y^2}{\omega^2} = A^2\left[\cos^2(\omega t - \alpha) + \frac{\omega^2}{\omega^2}\sin(\omega t - \alpha)\right]$$
$$= A^2 \checkmark$$

With small damping (b),

$$m\ddot{x} = -b\dot{x} - kx$$



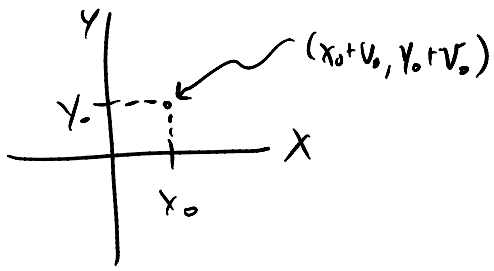
$x=y=0$ is an asymptotically stable C.P.

$$\frac{dx}{dt} = F(x,y)$$

$$\frac{dy}{dt} = G(x,y)$$

examine behavior of solns near critical pt $(x,y)=(x_0,y_0)$

$$\Rightarrow F(x_0,y_0) = G(x_0,y_0) = 0$$



Is (x_0,y_0) stable? Unstable? Asymptotically stable?

Let $x(t) = x_0 + u(t)$, where $|u_0| \ll 1$ $\quad u_0 = u(0)$

$y(t) = y_0 + v(t)$, where $|v_0| \ll 1$ $\quad v_0 = v(0)$

Taylor expand $F(x,y)$ & $G(x,y)$ about (x_0,y_0)

$$F(x_0+u, y_0+v) = \overbrace{F(x_0,y_0)}^{0 \text{ b/c } x_0,y_0 \text{ equilib}} + \frac{\partial F}{\partial x}(x_0,y_0)\,u + \frac{\partial F}{\partial y}(x_0,y_0)\,v + \cdots$$

$$f(x+h) = f(x_0) + f'(x_0)\,h + \tfrac{1}{2} f''(x_0)\,h^2 + \cdots$$

$$G(x_0+u, y_0+v) = \overbrace{G(x_0,y_0)}^{0} + \frac{\partial G}{\partial x}(x_0,y_0) + \frac{\partial G}{\partial y}(x_0,y_0)\,v + \cdots$$

$$\frac{dx}{dt} = \frac{d}{dt}(x_0+u) = \frac{du}{dt}, \qquad \frac{dy}{dt} = \frac{d}{dt}(x_0+v) = \frac{dv}{dt}$$

Linearized eqns of motion (neglect nonlinear terms in u, v)

$$\frac{du}{dt} = F_x(x_0,y_0)\,u + F_y(x_0,y_0)\,v$$

$$\frac{dv}{dt} = G_x(x_0,y_0)\,u + G_y(x_0,y_0)\,v$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{J(x_0,y_0)} \begin{pmatrix} u \\ v \end{pmatrix}$$

Jacobian

Example

Example

Jacobian

$$\frac{dx}{dt} = 3x - x^2 - xy$$

$$\frac{dy}{dt} = y + x^2 - 3xy$$

Find all C.P.'s & determine the linearized eqns. near each C.P.

Find C.P.

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \text{ at } (x_0, y_0)$$

$$3x - x^2 - xy = 0 \qquad x(3 - x - y) = 0$$

$$y + y^2 - 3xy = 0 \qquad y(1 + y - 3x) = 0$$

$$(x_0, y_0) = (0,0)$$

$$3 - x - y = 0$$

$$1 + y - 3x = 0$$

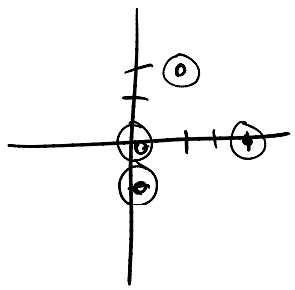
$$4 - 4x = 0$$

$$x = 1$$

$$(x_0, y_0) = (1, 2)$$

$$(x_0, y_0) = (0, -1)$$

$$(x_0, y_0) = (3, 0)$$

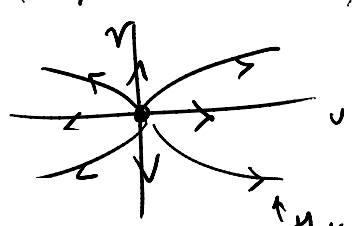


$$J(x,y) = \begin{bmatrix} f_x & f_y \\ G_x & G_y \end{bmatrix} = \begin{pmatrix} 3 - 2x - y & -x \\ -3y & 1 + 2y - 3x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

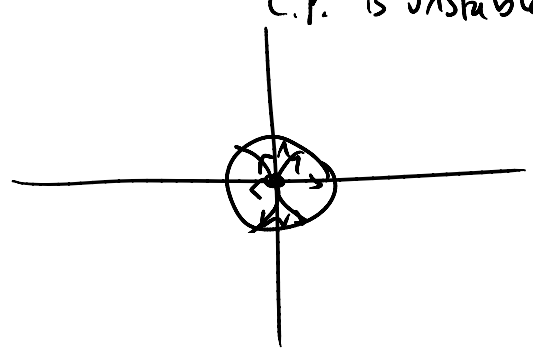
$$\frac{du}{dt} = 3u$$

$$\frac{dv}{dt} = v$$



this way b/c it's going faster in the u direction b/c of the 3u.

C.P. is unstable.



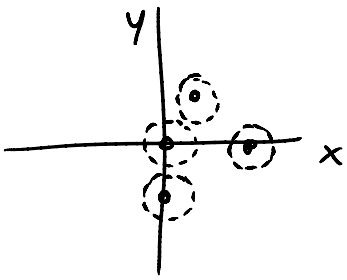
Linearized Eqns

Wednesday, May 23, 2007
9:03 AM

<u>Example</u>

$$\frac{dx}{dt} = 3x - x^2 - xy \qquad \frac{dy}{dt} = y + y^2 - 3xy$$

4 C.P.'s $(x,y) = (0,0), (0,-1), (3,0), (1,2)$



$$\frac{dx}{dt} = F(x,y) = F(x_0+u, y_0+v) =$$

$$\frac{dy}{dt} = G(x,y) = G(x_0+u, y_0+v)$$

C.P. $(x,y) = (x_0, y_0)$

let $\left.\begin{array}{l} x = x_0 + u \\ y = y_0 + v \end{array}\right\} |u|, |v| \ll 1$

$$= \cancelto{0}{F(x_0,y_0)} + \frac{\partial F}{\partial x}(x_0,y_0)u + \frac{\partial F}{\partial y}(x_0,y_0)v + \cancel{\ldots}$$

$$\cancelto{0}{G(x_0,y_0)} + \frac{\partial G}{\partial x}(x_0,y_0)u + \frac{\partial G}{\partial y}(x_0,y_0)v + \cancel{\ldots}$$

$$\frac{dx}{dt} = \frac{d}{dt}(x_0+u) = \frac{du}{dt}$$

$$\frac{dy}{dt} = \frac{d}{dt}(y_0+v) = \frac{dv}{dt}$$

u^2, v^2 etc much smaller than other #s u & v so ignore.

<u>Linearized Eqns</u>

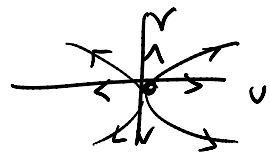
$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = J(x_0,y_0)\begin{pmatrix} u \\ v \end{pmatrix}$$

Jacobian matrix

$$J(x_0,y_0) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$

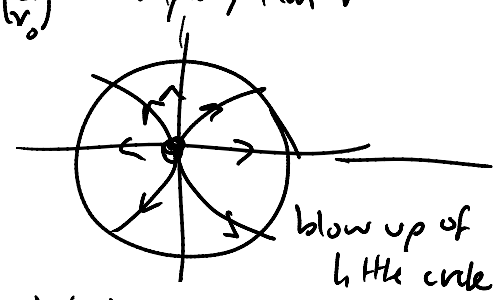
$$J = \begin{pmatrix} 3-2x-y & -x \\ -3y & 1+2y-3x \end{pmatrix}$$

$(0,0)$: $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$



$$\left.\begin{aligned} \frac{du}{dt} &= 3u \Rightarrow u = u_0 e^{3t} \\ \frac{dv}{dt} &= v \Rightarrow v = v_0 e^{t} \end{aligned}\right\} u = u_0 \left(\frac{v}{v_0}\right)^3$$

Inc. in u more rapidly than v



$(0,-1)$: $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\text{Det}\begin{pmatrix} 4-\lambda & 0 \\ 3 & -1-\lambda \end{pmatrix} = (4-\lambda)(-1-\lambda) = 0$$

$\lambda_1 = 4$

$\lambda_2 = -1$

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 e^{4t}\vec{v}_1 + c_2 e^{-t}\vec{v}_2$$

this is pos, so any non zero val. of c_1 will make soln blow up. So unstable c.p.

$(J - \lambda I)\vec{v}_1 = \vec{0}$

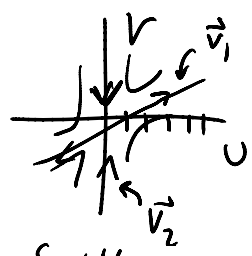
$$\begin{bmatrix} 0 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3w_1 - 5w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$(J - \lambda I)\vec{v}_2 = \vec{0}$

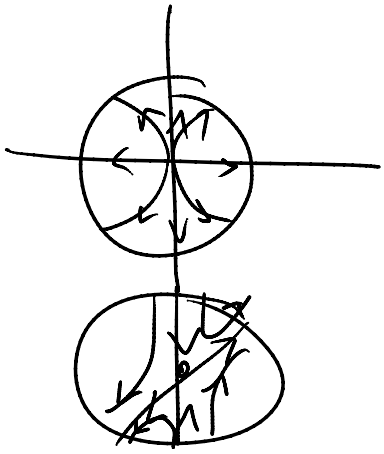
$$\begin{bmatrix} 5 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 5w_1 \\ 3w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Saddle pt

pt

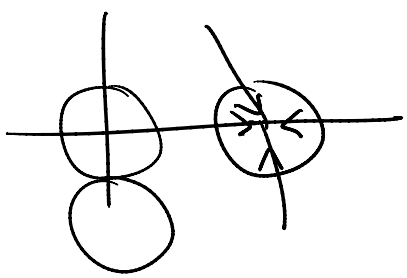


(3, 0)

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ 0 & -8 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 e^{-3t}\vec{v}_1 + c_2 e^{-8t}\vec{v}_2$$

This is stable. Goes to 0.



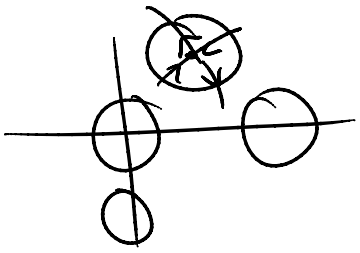
(1, 2) $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\operatorname{Det}\begin{pmatrix} -1-\lambda & -1 \\ -6 & 2-\lambda \end{pmatrix} = (-1-\lambda)(2-\lambda) - 6 = \lambda^2 - \lambda - 8 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+32}}{2}$$

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{33}}{2}$$

saddle pt



Lin. autonomous homogeneous eqn

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

$(x,y) = (0,0)$ is an isolated c.p.

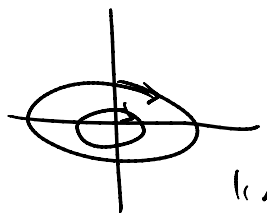
$(ad - bc \neq 0)$

Let λ_1, λ_2 = eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

If λ_1 & λ_2 have neg. real part $\rightarrow$ c.p. is asymptotically stable

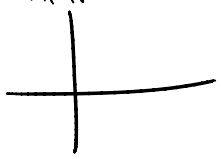
If either λ_1 or λ_2 has pos. real part, c.p. is unstable.

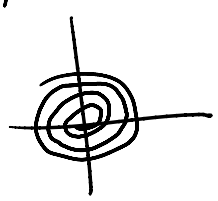
If $\lambda_1, \lambda_2 = \pm i\beta$ $(\beta \neq 0)$ stable but not asymptotically stable



linear prob w/ $\lambda_1, \lambda_2 = \pm i\beta$

nonlin.





slow spiraling

Phase Portraits

Friday, May 25, 2007
9:08 AM

Stability can often be determined by linearizing (*) about $(x,y_0) = (x_0, y_0)$

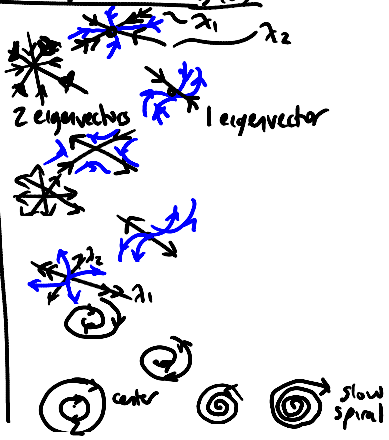
$$\frac{dx}{dt} = F(x,y) \qquad \frac{dy}{dt} = G(x,y)$$

$(x,y) = (x_0,y_0)$ is a c.p. if $F(x_0,y_0) = G(x_0,y_0) = 0$

$x = x_0 + u$
$y = y_0 + v$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = S(x_0,y_0)\begin{pmatrix} u \\ v \end{pmatrix} \qquad S = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}$$

Focus on eigenvalues of matrix about c.p.

eigenvalues of $S(x,y)$	Stability prop. of c.p.	Phase portrait near (x_0,y_0)
$\lambda_1 < \lambda_2 < 0$	asymptotically stable node	
$\lambda_1 = \lambda_2 < 0$	asymptotically stable node/ spiral	
$\lambda_1 < 0 < \lambda_2$	unstable saddle	
$\lambda_1 = \lambda_2 > 0$	unstable node / spiral	
$\lambda_1 > \lambda_2 > 0$	unstable node	
$\lambda_1, \lambda_2 = a \pm ib \ (a<0, b \neq 0)$	stable spiral	
$\lambda_1, \lambda_2 = a \pm ib \ (a>0, b \neq 0)$	unstable spiral	
$\lambda_1, \lambda_2 = \pm ib \ (b \neq 0)$	stable/unstable, center pt or spiral	

Phase portrait for the pendulum

θ = angle of pendulum to vertical

$\dot\theta$ = angular velocity

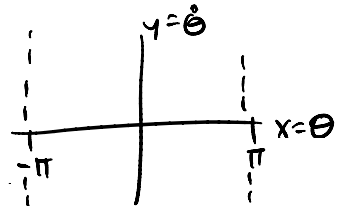
$\ddot\theta = -\frac{g}{l}\sin\theta$ = angular acceleration

small $\measuredangle$ approx $\rightarrow \sin\theta \approx \theta$

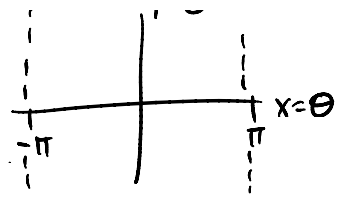
$$\ddot\theta = -\frac{g}{l}\theta \quad \leftarrow \left(\text{c.f. lin spring } \ddot{x} = -\frac{k}{m}x\right)$$

$\dot\theta = y$ = ang. velocity

$\dot{y} = -\frac{g}{l}\sin\theta$



phase space is a cylinder



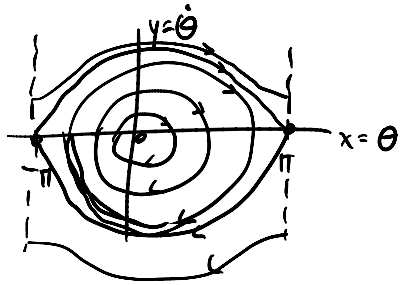
phase space is a cylinder

phase portrait from conservation of energy

$$\text{Energy} = KE + PE$$
$$= \left(\tfrac{1}{2} I \omega^2 = \tfrac{1}{2} m \ell^2 \dot\theta^2\right) + \left(mgh = mg(\ell - \ell\cos\theta)\right)$$

$$\frac{E}{m\ell^2} = \tfrac{1}{2}\dot\theta^2 + \frac{g}{\ell}(1-\cos\theta) = \tfrac{1}{2}y^2 + \frac{g}{\ell}(1-\cos\theta)$$

Soln curves are level surfaces of constant energy



Example probs

Tuesday, May 29, 2007
9:02 AM

$\vec{x}' = (\quad)\vec{x}$, $\lambda = -1, 3, 3$

$\vec{v}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$9a + 2b - 4c = 0$

$b = 0 \Rightarrow \begin{pmatrix} 4 \\ 3 \\ 9 \end{pmatrix}$

$a = 0 \Rightarrow \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$

$c = 0 \Rightarrow \begin{pmatrix} 2 \\ -9 \\ 0 \end{pmatrix}$

#30. 6.2)

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y-1 \\ x^2-y \end{pmatrix}$

Find c.p.'s ; $x' = 0$, $y' = 0$

$x' = 0$ $\quad y - 1 = 0 \Rightarrow y = 1$

$y' = 0$ $\quad x^2 - 1 = 0 \Rightarrow x = \pm 1$

c.p.'s $(-1, 1)$, $(1, 1)$

Stability:

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x & -1 \end{pmatrix}$$

$$J(-1, 1) = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

$A - \lambda I = 0$

$$\begin{bmatrix} -\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix} = 0$$

$\lambda(1+\lambda) + 2 = 0$

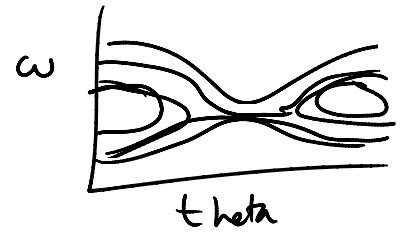
$\lambda^2 + \lambda + 2 = 0$

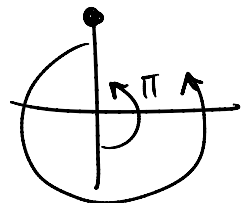
$\lambda = \frac{-1 \pm \sqrt{1-8}}{2}$

Stable spiral

Stable spiral

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$





Example probs
Wednesday, May 30, 2007
9:03 AM

$x' = 2x + 3y + 5 \qquad y' = 2x + y - 2t$

$$\vec{x}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 5 \\ -2t \end{bmatrix}$$

$$\vec{x}_h = A \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} e^{4t} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

(soln to $\vec{x}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \vec{x}$ eigenval prob

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

$$\vec{x}_p = \begin{bmatrix} a+bt \\ c+dt \end{bmatrix}$$

$$\vec{x}_p' = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a+bt \\ c+dt \end{bmatrix} + \begin{bmatrix} 5 \\ -2t \end{bmatrix}$$

4 eqns for a, b, c, d
equate coefficients in front of same types of terms

$$\begin{bmatrix} 2a + 2bt + 3c + 3dt + 5 \\ 2a + 2bt + c + dt - 2t \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 & 0 \\ 2 & 0 & 1 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

t terms must cancel out

$2bt\,d - 2 = 0$

$2bt + 3dt = 0$

$$\vec{x}_{n+1} = \vec{x}_n + \frac{h}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + k_4)$$

where $\vec{k}_1 = \vec{f}(t_n, \vec{x}_n)$, $\vec{k}_2 = (t_n + \frac{h}{2}, \vec{x}_n + \frac{h\vec{k}_1}{2})$

$\vec{k}_3 = \vec{f}(t_n + \frac{h}{2}, \vec{x}_n + \frac{h\vec{k}_2}{2})$, $\vec{k}_4 = \vec{f}(t_{n+1}, \vec{x}_n + h\vec{k}_3)$

$y'' + y^2 = t \qquad y(0) = 0, \quad y'(0) = 0$

1st step of Runga Kutta, determine y at next step, $y_1 = y(h)$.

$y'' + y^2 = t$

$w = y'$

$w = y'$

$w' = y'' = t - y^2$

$$\begin{bmatrix} y' \\ w' \end{bmatrix} = \begin{bmatrix} w \\ t - y^2 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} y \\ w \end{bmatrix}, \vec{x}' = \begin{bmatrix} x_2 \\ t - x_1^2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

Plug into Runga Kutta formulas

$$\vec{x}' = \vec{f}(t, \vec{x})$$

$$\vec{f} = \begin{bmatrix} x_2 \\ t - x_1^2 \end{bmatrix} \qquad \begin{matrix} y(0) = 0 \\ y'(0) = 0 \end{matrix} \Rightarrow \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

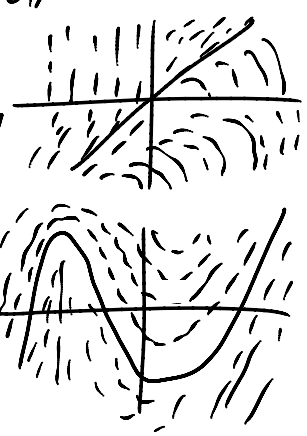
$$\vec{k}_1 = \vec{f}(t_n + \vec{x}_n) \quad n = 0$$
$$= f\left(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{k}_2 = \vec{f}\left(t_n + \frac{h}{2}, x_n + \frac{h\vec{k}_1}{2}\right) = \vec{f}\left(\frac{h}{2}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \frac{h}{2} \end{bmatrix}$$

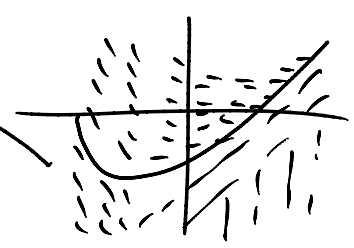
Match direction fields to eqns

$$\frac{dy}{dx} = x^2 - y - 2$$

$$\frac{dy}{dx} = y - x + 1$$



$$\frac{dy}{dx} = x - y$$

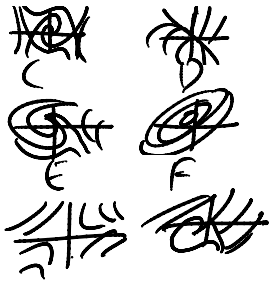


$$\frac{dy}{dx} = x + y$$

$$\frac{dx}{dt} = x - y \qquad \frac{dy}{dt} = x + 3y - 4$$

C.P. (1,1)

A B



D

$\frac{dx}{dt} = 1 - y^2 \quad \frac{dy}{dt} = x + 2y$

C.P $(-2,1)$ $(2,-1)$

A

$\frac{dx}{dt} = 2x - 2y - 4 \quad \frac{dy}{dt} = x + 4y + 3$

C.P $(1,-1)$ B

$\frac{dx}{dt} = 2x - y \quad \frac{dy}{dt} = x - 3y$

C

Solve: $\frac{dx}{dt} = x(5 - x), \quad x(0) = 3$

Ans $\dfrac{\frac{15}{2}e^{5t}}{1 + \frac{3}{2}e^{5t}} = x$

$xy' + 5y = 7x^2, \quad y(1) = 5$

$y = x^2 + \frac{4}{x^5}$

$y'' - 4y = 2e^{2x}$

$y_1 = Ae^{2x} + Be^{-2x}$

$y_p = Cxe^{2x}$

$y = Ae^{2x} + Be^{-2x} + \frac{1}{2}xe^{2x}$

$x'' + 4x = 5\sin(3t) \quad x(0) = x'(0) = 0$

$x = \frac{3}{2}\sin 2t - \sin 3t$

$x' = 2x + 3y + 5 \quad y' = 2x - y - 2t$

$x_h = A\begin{pmatrix}1\\ \frac{2}{3}\end{pmatrix}e^{4t} + B\begin{pmatrix}1\\ -1\end{pmatrix}e^{-t}$

$x_p = \begin{pmatrix}\frac{1}{8} + \frac{3}{2}t\\ -\frac{5}{4} - t\end{pmatrix}$

$x = x_h + x_p$

$\frac{dx}{dt} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\vec{x} \quad ; \quad \vec{x}(0) = \begin{bmatrix}1\\ 1\end{bmatrix}$

$\lambda = \{1, 1\}$

$\vec{x} = \begin{bmatrix}1\\ 0\end{bmatrix}e^{t} + \begin{bmatrix}t\\ 1\end{bmatrix}e^{-t}$

$\vec{x}' = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\vec{x}$

$(1-\lambda)(1-\lambda) = 0$

$\lambda = 1$

$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow$

$x = \begin{bmatrix}1\\ 0\end{bmatrix}e^{t}$

$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix}0\\ 1\end{bmatrix}$

$\Rightarrow x = \left(\begin{bmatrix}1\\ 0\end{bmatrix}t + \begin{bmatrix}0\\ 1\end{bmatrix}\right)e^{t}$

$x = A\begin{bmatrix}1\\ 0\end{bmatrix}e^{t} + B\begin{pmatrix}t\\ 1\end{pmatrix}e^{t}$

$\vec{x}' = A\vec{x}$

$Av_1 = \lambda v_1 \Rightarrow (A - \lambda I)\vec{v}_1 = 0$

$Av_2 = \lambda v_2 + v_1$

$\Rightarrow (A - \lambda I)\vec{v}_2 = v_1$

Practice final

Friday, June 01, 2007

9:03 AM

Practice Final

(4) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} -\frac{1}{10} & 0 & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{10} & 0 \\ 0 & \frac{1}{10} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$

(5) Variation of Parameters ... $u_1 = \int \sin x \, dx$, $u_2 = \int -\frac{\sin^2 x}{\cos x} dx$

(6) $\vec{\lambda}_{n+1} = \begin{bmatrix} h^2/2 - h^4/46 \\ h^2/6 \end{bmatrix}$

(7) (2, 3) stable spiral

(0, 0) saddle

(5, 0) saddle pt

long term behavior $\longrightarrow$ 20,000 prey, 30,000 predator