

Multiple Integration

Wednesday, January 03, 2007

10:03 AM

Multiple Integration

$$\iint_R f(x, y) dx dy$$

$$\iiint_V g(x, y, z) dx dy dz$$

Midterm after this stuff

2nd half is vector calculus

David Hansen is the new TA

Multiple Integration

Interpreting double & triple integrals

Techniques for these integrals

Developing applications

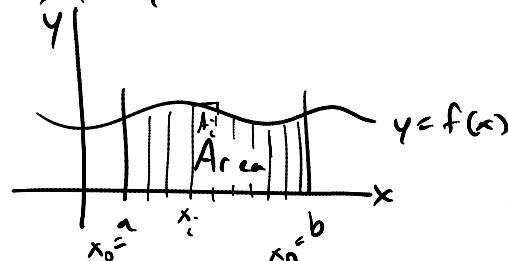
Calculating center of mass

Calculating moments of inertia

Integrals of a single variable & calculating the area under a curve

Riemann Sums

In particular, interpretation in terms of area under a curve



$$\text{Area} = \lim_{\Delta x \rightarrow 0} \sum_{i=0}^n f(x_i) \Delta x = \int_a^b f(x) dx$$

$$A_i \approx \Delta x \cdot f(x_i)$$

$$\Delta x = \frac{b-a}{n}$$

$$A_i = \left(\frac{b-a}{n}\right) f(x_i)$$

$$\text{Area} = \sum_{i=1}^n A_i \approx \sum_{i=0}^n f(x_i) \Delta x$$

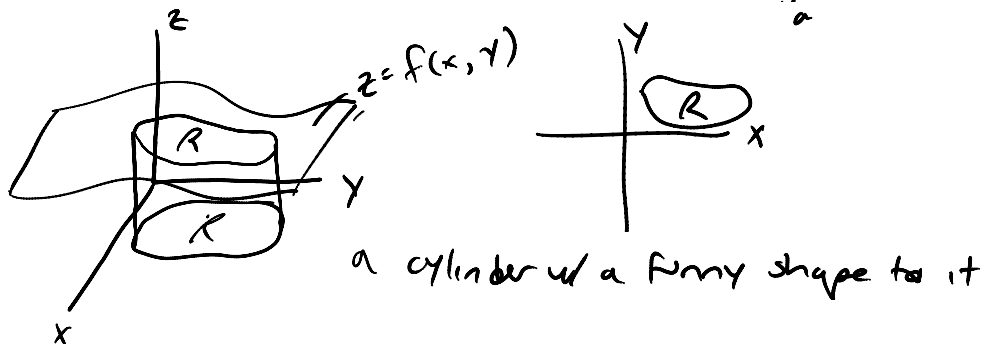
b/c left endpt.

Do the exact same thing for a double integral.

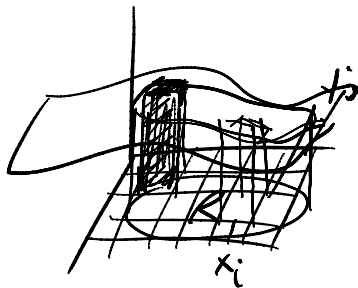
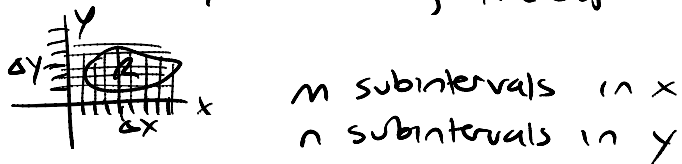
$$\iint_R f(x, y) dx dy$$

interpret volume between (x, y) plane & the surface $z = f(x, y)$

σ inside the region R (117) f
 R takes the place of a, b in $\int_a^b$



Estimate the volume by subdividing the object into smaller objects



Treat it as if the top is flat.

Volume V_{ij} = volume of (i, j) rectangular prism
 $\approx \Delta x \Delta y f(x_i, y_j)$

$$\text{Volume} = \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \Delta x \Delta y$$

$$\text{Volume} \approx \sum_{(i,j)} f(x_i, y_j) \Delta x \Delta y$$

You can get a better & better idea of the volume by dividing into smaller intervals

$$= \lim_{\Delta x \Delta y \rightarrow 0} \sum_{i,j} f(x_i, y_j) \Delta x \Delta y$$

$\iint_R f(x, y) dx dy$

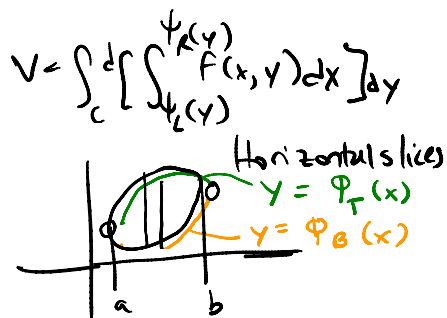
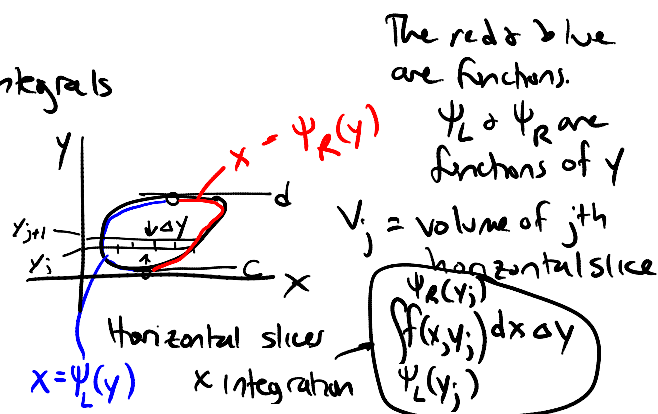
If $f(x,y) = 1$, then what is $\iint_R dx dy$?

$$\iint_R dx dy = \text{area of } R$$

Evaluation of double integrals

Iterated Integrals

$$\iint_R f(x,y) dx dy$$



$$\int_a^b \left[\int_{\phi_B(x)}^{\phi_T(x)} f(x,y) dy \right] dx$$

Set up as a usual integral where we treat y like it's always at y_i

$$\begin{aligned} V &\approx \sum_{j=1}^n \int_{\psi_L(y_j)}^{\psi_R(y_j)} f(x, y_j) dx \Delta y \\ &= \lim_{\Delta y \rightarrow 0} \sum_{j=1}^n \int_{\psi_L(y_j)}^{\psi_R(y_j)} f(x, y_j) dx \Delta y \\ &= \int_c^d \left[\int_{\psi_L(y)}^{\psi_R(y)} f(x, y) dx \right] dy \end{aligned}$$

The ψ & ϕ functions determine first do that $[]$, then where the x_i & y_i is at that moment $\int_c^d dy$

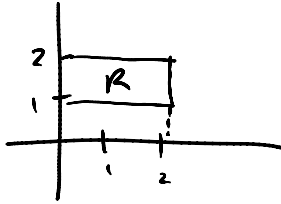
Examples

Thursday, January 04, 2007
9:57 AM

Ex 1

$$\iint_R 3xy^2 dx dy$$

$$R \quad 0 \leq x \leq 2 \\ 1 \leq y \leq 2$$



a) vertical strips



b) Horizontal strips



$$a) \int_0^2 \left[\int_1^2 3xy^2 dy \right] dx$$

x is a constant for the 1st integral.

$$3x \int_1^2 y^2 dy$$

$$3x \left[\frac{1}{3} y^3 \right]_1^2 = 7x$$

$$\int_0^2 7x dx$$

$$7 \left[\frac{x^2}{2} \right]_0^2 = 14$$

$$b) \int_1^2 \left[\int_0^2 3xy^2 dx \right] dy$$

y is now a constant

$$3y^2 \int_0^2 x dx$$

$$3y^2 \left[\frac{x^2}{2} \right]_0^2 = 6y^2$$

$$\int_1^2 6y^2 dy = 6 \left[\frac{y^3}{3} \right]_1^2 = 14$$

$$\frac{2(2^3) - 2(1^3)}{16 - 2} = 14$$

Ex 2

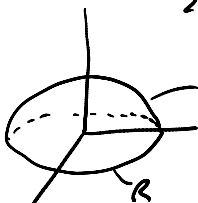
Compute the volume of the hemisphere

$$x^2 + y^2 + z^2 \leq 1$$

$$z \geq 0$$

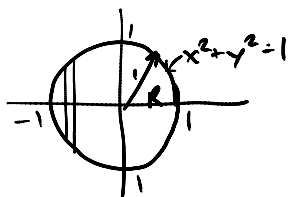
$$V = \frac{2}{3} \pi$$

$$= \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right), \text{ radius } = \frac{2}{3} \pi$$



$$z = f(x, y) = \sqrt{1 - x^2 - y^2}$$

$$V = \iint_R f(x, y) dx dy$$



Vertical strips or horizontal. It doesn't matter.

$$V = 2 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx$$

$$\sqrt{a^2 - y^2}, a^2 = 1 - x^2$$

$$\frac{1}{4} \pi (1 - x^2)$$

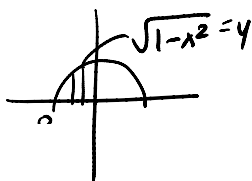
by geometry

$$is = \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dy$$

The 2 b/c the top & bottom are the same

If they weren't symmetrical, it'd be

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$



Directly

$$\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} dy$$

Try substitution

$$let y = \sqrt{1-x^2} \sin \theta$$

$$dy = \sqrt{1-x^2} \cos \theta d\theta$$

$$1-x^2-y^2 = (\sqrt{1-x^2})(\sqrt{1-\sin^2 \theta}) = \sqrt{1-x^2} \cos \theta$$

Think of

x as a constant

radius times sin

$$\begin{aligned} 1-x^2-y^2 &= 1-x^2-(1-x^2)\sin^2 \theta \\ &= (1-x^2)(1-\sin^2 \theta) \\ &= (1-x^2)(\cos^2 \theta) \end{aligned}$$

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} (1-x^2) \cos^2 \theta d\theta \\ &\quad \underbrace{\qquad\qquad\qquad}_{\frac{1}{2}(1+\cos 2\theta)} \end{aligned}$$

$$\begin{aligned} &(1-x^2) \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 + \cos 2\theta d\theta \\ &\quad \left[\frac{1}{2} (1-x^2) \left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^{\frac{\pi}{2}} \\ &\quad \frac{\pi}{4} (1-x^2) \end{aligned}$$

$$\begin{aligned} V &= 2 \int_{-1}^1 \frac{\pi}{4} (1-x^2) dx = \frac{\pi}{2} \left(x - \frac{x^3}{3} \right) \Big|_{-1}^1 \\ &= \frac{\pi}{2} \left(\left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) \right) \\ &= \frac{\pi}{2} \left(\frac{2}{3} - \left(-\frac{2}{3} \right) \right) = \frac{\pi}{2} \left(\frac{4}{3} \right) = \frac{2\pi}{3} \end{aligned}$$

Ex 3

Evaluate $I = \int_0^1 \int_0^1 e^{-x^2} dx dy$ This is really, really hard.

(BJT) if you reverse the order of integration...

$$= \iint_R e^{-x^2} dy dx$$

Step 1

... what R is

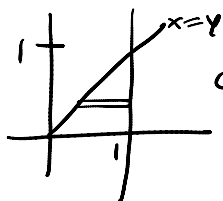
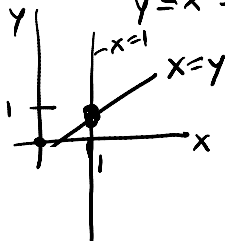
Step 1

Figure out what R is.

$$0 \leq y \leq 1$$

$$y \leq x \leq 1$$

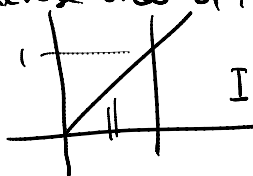
From the boundaries above



Common sense tells you which Δ it is

Step 2

Reverse order of integration.



$$I = \int_0^1 \int_0^x e^{-x^2} dy dx$$

$$e^{-x^2} y \Big|_0^x = x e^{-x^2}$$

$$I = \int_0^1 x e^{-x^2} dx$$

$$I = \left[-\frac{1}{2} e^{-x^2} \right]_0^1 = -\frac{1}{2} e^{-1} + \frac{1}{2} e^0$$

$$= \frac{1}{2} \left(1 - \frac{1}{e} \right)$$

Integration by parts returns!

Trick: $I = \int_0^1 \int_y^1 e^{-x^2} dx dy$

directly

$$\int u dv = uv - \int v du$$

$$\int_0^1 \underbrace{\int_y^1 e^{-x^2} dx}_{f(y)} dy =$$

$$\int_0^1 \underbrace{f(y)}_u \underbrace{dy}_{dv} = \underbrace{f(y)y}_v \Big|_0^1 - \int_0^1 \underbrace{f'(y)}_{\text{where } f(y) = \int_y^1 e^{-x^2} dx} dy$$

$$u = f(y) \quad du = f'(y) dy$$

$$dv = dy \quad v = y$$

$$\int_0^1 f'(y) y dy$$

$$f'(y) = \frac{d}{dy} \left[-\int_y^1 e^{-x^2} dx \right] = -e^{-y^2}$$

$$\int_0^1 e^{-y^2} dy$$

Volume of Intersection

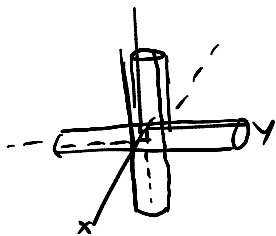
Friday, January 05, 2007

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Compute the volume of intersection between the cylinders

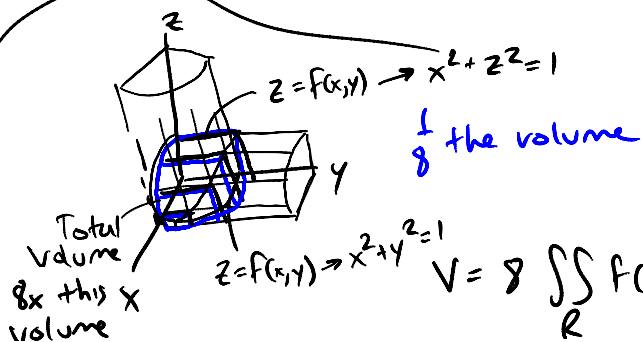
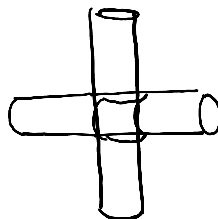
$$x^2 + y^2 = 1$$

circling z axis.

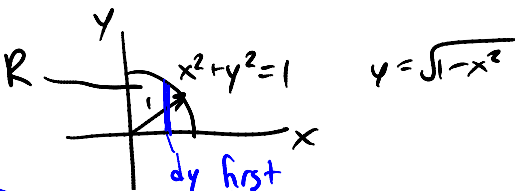


$$x^2 + z^2 = 1$$

circling y axis



p 960



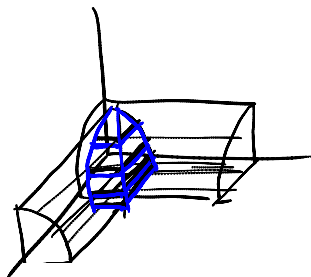
$$z = f(x, y) = \sqrt{1 - x^2}$$

$$V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} \, dy \, dx$$

$$(\sqrt{1-x^2}) y \Big|_0^{\sqrt{1-x^2}} = 1 - x^2$$

$$V = 8 \int_0^1 (1 - x^2) \, dx$$

$$V = 8 \left(x - \frac{x^3}{3} \right) \Big|_0^1 = \left(\frac{16}{3} \right)$$



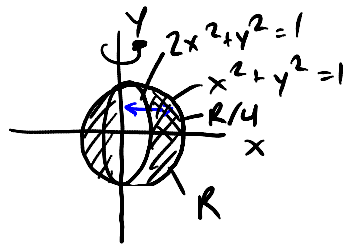
Ex

Compute moment of inertia around the y axis of a lamina (thin plate) in the shape of the region R, between $x^2 + y^2 = 1$ and $2x^2 + y^2 = 1$. Assume density $\delta = 1 \text{ kg/cm}^2$. Note: Actual density is $\rho = \frac{\delta}{t}$ where $t = \text{thickness}$

But thickness is constant so it's not cm^3 , it's cm^2 .

$$I = \iint_R \delta d^2 dx dy$$

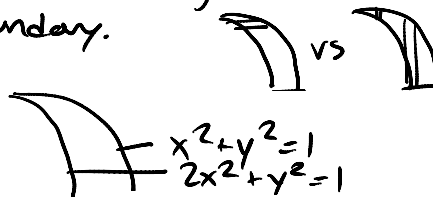
(mass) at (x, y)
distance from material element to axis of rotation.



$\delta = 1$ $d = x$ b/c it varies w/ x

Horizontal strips b/c single formula for inner/outer boundary.

$$I = 4 \int_0^1 \int_{\frac{\sqrt{1-y^2}}{2}}^{\sqrt{1-y^2}} x^2 dx dy$$



$$I = 4 \int_0^1 \frac{x^3}{3} \Big|_{\frac{\sqrt{1-y^2}}{2}}^{\sqrt{1-y^2}} dy$$

$$x = \sqrt{1-y^2}$$

$$x = \frac{\sqrt{1-y^2}}{2}$$

$$I = \frac{4}{3} \int_0^1 (1-y^2)^{\frac{3}{2}} \left(1 - \frac{1}{2\sqrt{2}}\right) dy$$

Trig substitution, since
let $y = \sin \theta$
 $dy = \cos \theta$

$$I = \frac{4}{3} \int_0^1 (1-\sin^2 \theta)^{\frac{3}{2}} \left(1 - \frac{1}{2\sqrt{2}}\right) \cos \theta d\theta$$

$$I = \frac{4}{3} \left(1 - \frac{1}{2\sqrt{2}}\right) \int_0^{\frac{\pi}{2}} (1-y^2)^{\frac{3}{2}} dy$$

$$I = \frac{4}{3} \left(1 - \frac{1}{2\sqrt{2}}\right) \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta$$

$\frac{3}{16} \pi$

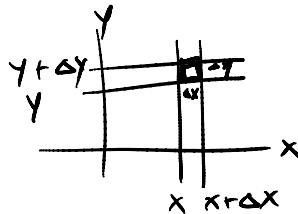
Convert Double Integrals in Cartesian Coordinates

$$\iint_R f(x, y) dx dy \rightarrow \text{to polar coordinates}$$

$\downarrow \quad \downarrow$
 $r \cos \theta \quad r \sin \theta$
 $dA \rightarrow \text{Area of an infinitesimal } \Delta$

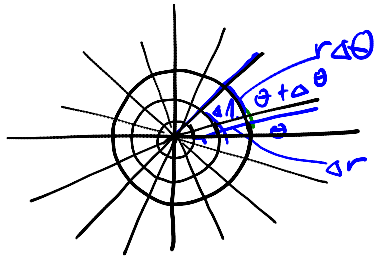
$$r \cos \theta = x, r \sin \theta = y$$

$$\iint_R f(x, y) dA$$



$$\Delta A = \Delta x \Delta y$$

$$dA = dx dy$$



$$\Delta A \approx (r \Delta \theta) \Delta r$$

$$dA = r d\theta dr$$

Same thing but polar.

not a real rectangle but if you let $\Delta r \rightarrow 0$, it becomes exact.

Fraction of the full circle to find Area of sector.

$$\Delta A = \left(\frac{\Delta \theta}{2\pi}\right) \pi (r + \Delta r)^2 - \left(\frac{\Delta \theta}{2\pi}\right) \pi r^2$$

$$\Delta A = \frac{\Delta \theta}{2} [r^2 + 2r\Delta r + \Delta r^2 - r^2]$$

$$\Delta A = r\Delta r\Delta \theta + \frac{\Delta \theta}{2} \Delta r^2$$

$$\lim_{\Delta r, \Delta \theta \rightarrow 0} \Delta A = r\Delta r\Delta \theta + \frac{\Delta \theta}{2} \Delta r^2 = \Delta r\Delta \theta \left[r + \frac{\Delta r}{2} \right]$$

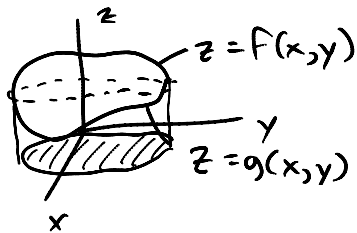
$$dA = dr d\theta r$$

much smaller factor, negligible

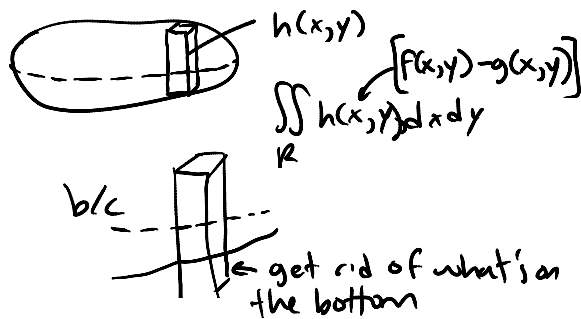
$$\iint_R f(x, y) \underbrace{dx dy}_{r dr d\theta} = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta$$

Shadow, change coordinate sys

Monday, January 08, 2007
10:02 AM

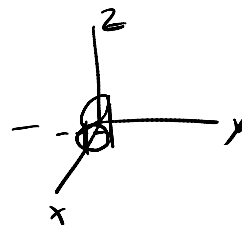
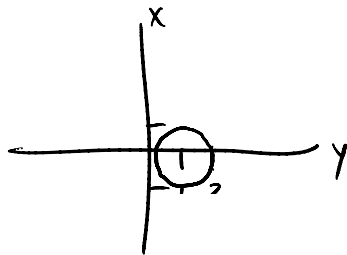


b/c they meet at z
 $f(x, y) = g(x, y)$



Ex

Compute the volume between $z=0$ and $z=2x+3y+2$ above the region defined by $(x-1)^2 + y^2 \leq 1$



Polar coordinates (b/c R is a $\odot$)

$$V = \iint_R (2x + 3y + 2) dx dy$$

$\downarrow$ $\downarrow$ $\downarrow$
 R $r \cos \theta$ $r \sin \theta$ $r dr d\theta$

$$\begin{aligned} (x-1)^2 + y^2 &= 1 \\ x^2 - 2x + 1 + y^2 &= 1 \\ x^2 - 2x + y^2 &= 0 \\ r^2 \cos^2 \theta - 2r \cos \theta + r^2 \sin^2 \theta &= 0 \\ r^2 - 2r \cos \theta &= 0 \end{aligned}$$

$r = 2 \cos \theta$ the $\odot$ in polar coordinates.
 $0 \leq \theta \leq \pi$ b/c you're iterating twice
($0 - 2\pi$ is the same as $0 - \pi$, since it's a $\odot$)

$$z = 2x + 3y + 2$$

$$\begin{aligned} V &= \int_0^{\pi} \int_0^{2 \cos \theta} (2r \cos \theta + 3r \sin \theta + 2) r dr d\theta \\ V &= \int_0^{\pi} \left[\frac{2}{3} r^3 \cos \theta + \frac{3}{2} r^2 \sin \theta + 2r \right]_0^{2 \cos \theta} d\theta \end{aligned}$$

if it's $z = 2x + 3y + 1$,
it intersects the $\odot$
has a pos + neg volume.



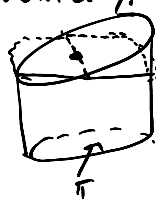
equal to
if 2π , you trace
the circle 2 times.

$$\begin{aligned}
 V &= \int_0^\pi \left[\frac{2}{3} r^3 \cos \theta + r^3 \sin \theta + r^2 \right]_0^{2 \cos \theta} d\theta \\
 V &= \int_0^\pi \left[\frac{16}{3} \cos^4 \theta + 8 \cos^3 \theta \sin \theta + 4 \cos^2 \theta \right] d\theta \\
 V &= \int_0^\pi 8 \cos^3 \theta \sin \theta d\theta + \frac{16}{3} \int_0^\pi \cos^4 \theta d\theta + 4 \int_0^\pi \cos^2 \theta d\theta \\
 &\quad \begin{array}{l} u = \cos \theta \\ du = -\sin \theta d\theta \end{array} \quad \begin{array}{l} + \frac{16}{3} \int_0^\pi (\cos^2 \theta)^2 d\theta \\ + \frac{16}{3} \int_0^\pi \left(\frac{1}{4} (1 + 2 \cos 2\theta + \cos^2 2\theta) \right) d\theta \\ + \frac{16}{3} \int_0^\pi \left(\frac{1}{4} (1 + 2 \cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta) \right) d\theta \\ + \frac{16}{12} \int_0^\pi \left(\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta \end{array} \quad \begin{array}{l} + \frac{4}{2} \int_0^\pi (1 + \cos 2\theta) d\theta \\ \text{average value of this function is 0 when going from 0 to } \pi \end{array} \\
 V &= 8 \int_1^{-1} u^3 du \\
 V &= -8 \left[\frac{u^4}{4} \right]_1^{-1} \\
 V &= 0 \\
 &\quad + \frac{16}{3} \left(\frac{3}{8} \pi \right) \\
 &\quad + \frac{16}{12} \left(\frac{3}{2} \pi + 2 \pi \right) \\
 &\quad = \frac{4}{2} \left(\frac{1}{2} \theta \right) \Big|_0^\pi \\
 &\quad = 2\pi \\
 2\pi + 2\pi &= (4\pi)
 \end{aligned}$$

We've sliced a cylinder w/ a plane. We can find the volume of the Tokra crystal



geometrically.



you can make a new cylinder

evaluate function at center of $\odot$

$$z = 2x + 3y + 2$$

$(x, y) = (1, 0)$ b/c that's the center.

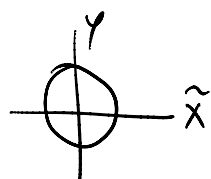
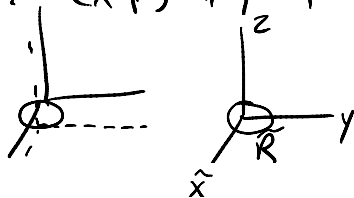
circle is of radius 1, so Area = π

If the $\odot$ was centered at the origin, it would have been really easy.

Change coordinate system to make it easier.

$$z = 2x + 3y + 2$$

$$R: (x-1)^2 + y^2 = 1$$



$$\hat{x} = x - 1$$

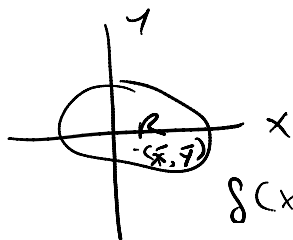
$$\hat{R} = \hat{x}^2 + y^2 = 1$$

$$z = 2(\hat{x} + 1) + 3y + 2$$

$$z = 2\hat{x} + 3y + 4$$

$$\iint_R (2\hat{x} + 3y + 4) d\hat{x} dy$$

$$= \int_0^{2\pi} \int_0^1 (2 \cos \theta + 3 \sin \theta + 4) r dr d\theta$$



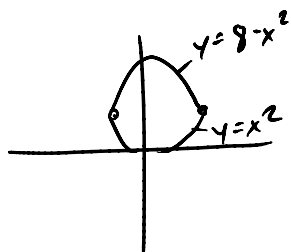
$\delta(x, y)$ = density in units
mass/area

$$\begin{aligned}
 &= \int_0^1 \int_0^{2\pi} (2r^2 \cos \theta + 3r^2 \sin \theta + 4r) r \, d\theta \, dr \\
 &= \int_0^1 4r \theta \Big|_0^{2\pi} \, dr \\
 &= 8\pi \int_0^1 r \, dr \\
 &= 8\pi \left[\frac{r^2}{2} \right]_0^1 = 4\pi
 \end{aligned}$$

$(\bar{x}, \bar{y})$ center of mass (no net torque at that point)
gravity completely balanced in x & y direction.

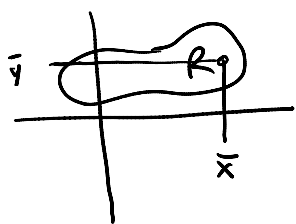
Center of Mass, Torque

Tuesday, January 09, 2007
9:00 AM



$$\iint_R y \, dA$$

$$\int_{-2}^2 \int_{x^2}^{8-x^2} y \, dy \, dx$$



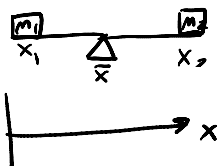
center of mass $(\bar{x}, \bar{y})$

Review

Torque

Angular momentum

Center of mass



Angular momentum about a pt.

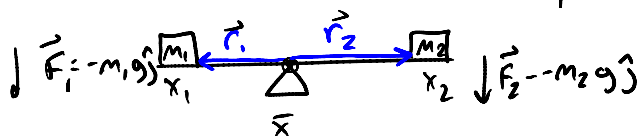
is $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{p} = m\vec{v}$,
linear momentum
& $\vec{r}$ = position vector

Torque is $\frac{d\vec{L}}{dt}$, about some pt.

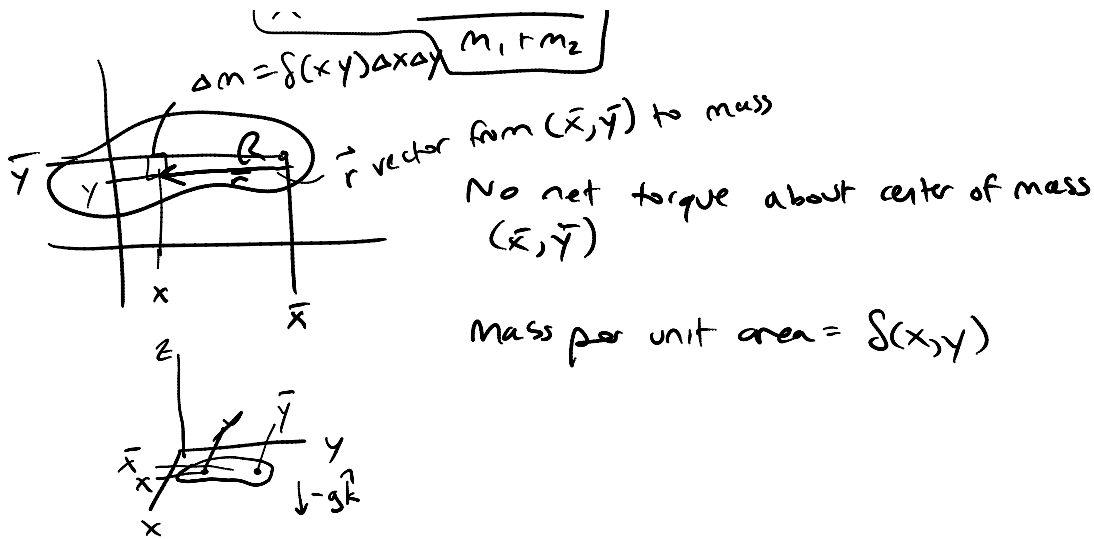
$$\begin{aligned} \vec{r}_1 &= (\bar{x} - x_1)\hat{i} = (x_1 - \bar{x})\hat{i} \\ \vec{r}_2 &= (x_2 - \bar{x})\hat{i} \end{aligned} \quad \left| \quad \vec{L} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \right.$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

= 0 + $\vec{r} \times \vec{F}$
bc velocity & linear momentum pt. in same direction.



$$\begin{aligned} \vec{\tau} &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 & \vec{\tau} &= \vec{r} \times \vec{F} \\ \vec{\tau} &= (x_1 - \bar{x})(-m_1 g)(\hat{i} \times \hat{j}) + (x_2 - \bar{x})(-m_2 g)(\hat{i} \times \hat{j}) \\ 0 &= kg [m_1 \bar{x} - m_1 x_1 + m_2 \bar{x} - m_2 x_2] \\ \vec{\tau} &= \bar{x} (m_1 - m_2) - (m_1 x_1 + m_2 x_2) = 0 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$



$$\Delta \vec{F} = \Delta m (-g \hat{k})$$

$$\vec{r} = \langle x, y, 0 \rangle - \langle \bar{x}, \bar{y}, 0 \rangle$$

$$\Delta \vec{\tau} = \vec{r} \times \Delta \vec{F} \leftarrow \text{contribution to torque by mass elmt}$$

$$= \langle x - \bar{x}, y - \bar{y}, 0 \rangle \times [\delta(x, y) \Delta A g \hat{k}]$$

$$\text{Total torque} = \vec{\tau} = g \iint_R \langle y - \bar{y}, x - \bar{x}, 0 \rangle \delta(x, y) dA$$

We want this = to 0

$$\iint_R \delta(x, y) (y - \bar{y}) dA = 0$$

$$\iint_R \delta(x, y) (x - \bar{x}) dA = 0$$

$$\left. \begin{array}{l} \iint_R \delta(x, y) (y - \bar{y}) dA = 0 \\ \iint_R \delta(x, y) (x - \bar{x}) dA = 0 \end{array} \right\} \begin{array}{l} \text{solve for} \\ (\bar{x}, \bar{y}) \\ \text{center of mass} \end{array}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x - \bar{x} & y - \bar{y} & 0 \\ 0 & 0 & -g \end{vmatrix}$$

$$= (y - \bar{y})(-g)\hat{i} - ((x - \bar{x})(-g))\hat{j} + 0\hat{k}$$

$$\iint_R \delta(x, y) y dA = \iint_R \delta(x, y) \bar{y} dA \quad \leftarrow \text{just a \#}$$

$$= \bar{y} \iint_R \delta(x, y) dA \quad \leftarrow \text{mass/Area integrated over Area.}$$

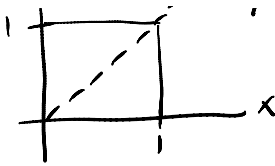
$M = \text{total mass}$

$$\bar{y} = \frac{1}{M} \iint_R \underbrace{\delta(x, y) dA}_{dm}$$

$$\bar{x} = \frac{1}{M} \iint_R \underbrace{x \delta(x, y) dA}_{dm}$$

$$M = \iint_R \underbrace{\delta(x, y) dA}_{dm}$$





$\delta(x,y) = x+y$ so max is 2
center of mass on line $y=x$; $\bar{x} = \bar{y}$

$$\frac{1}{M} \iint_R (x+y) \times dA$$

$$M = \iint_R (x+y) dA = \int_0^1 \int_0^1 (x+y) dx dy$$

$$\left[\frac{x^2}{2} + xy \right]_0^1 = \frac{1}{2} + y$$

$$\int_0^1 \left(\frac{1}{2} + y \right) dy$$

$$= \left[\frac{1}{2}y + \frac{y^2}{2} \right]_0^1 = 1$$

$$\bar{x} = \iint_R x(x+y) dx dy$$

$$\left[\frac{x^3}{3} + y \frac{x^2}{2} \right]_0^1 = \frac{1}{3} + \frac{y}{2}$$

$$\int_0^1 \left(\frac{1}{3} + \frac{y}{2} \right) dy$$

$$= \left[\frac{1}{3}y + \frac{1}{4}y^2 \right]_0^1 = \frac{7}{12}$$

$$(\bar{x}, \bar{y}) = \left(\frac{7}{12}, \frac{7}{12} \right)$$

b/c $\bar{x} = \bar{y}$

M 455

W 11-1300
 Th 1500-1700

d-hansen@n...

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{2-x^2-y^2} dy dx$$

$$\sqrt{2-x^2} \sqrt{1-\frac{y^2}{2-x^2}}$$

$$\int_0^1 \int_{y_1}^{y_2} \sqrt{a} \sqrt{1-\frac{y^2}{a}} dy dx$$

$$y = \sqrt{a} \sin u$$

$$dy = \sqrt{a} \cos u du$$

$$a \cos u \sqrt{1-}$$



$$\sqrt{a-y^2}$$

$$\frac{y}{\sqrt{a}} = \sin u$$

Polar

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{2-x^2-y^2} dy dx$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\int_0^1 \int_0^{\sqrt{1-r^2 \cos^2 \theta}} \sqrt{2-r^2 \cos^2 \theta - r^2 \sin^2 \theta} r dr d\theta - \int_0^1 \int_0^{\sqrt{1-r^2 \cos^2 \theta}} r^2 r dr d\theta$$

$$\int_0^1 \int_0^{\sqrt{1-r^2 \cos^2 \theta}} (\sqrt{2-r^2} r dr d\theta$$

$$2-r^2 = u$$

$$-2r dr = du$$

$$z = x^2 + 2y^2 \quad z = 12 - 2x^2 - y^2$$

$$\text{combine, } y = \sqrt{4-x^2}$$

$$\iint (12 - 2x^2 - y^2) - (x^2 + 2y^2) dx dy$$

$$z|_{y=0} = 12 - 2x^2 \text{ is just a parabola.}$$

$$z|_{x=0} = 12 - y^2$$

$$44. \quad z = x^2 + 3y^2 \quad z = 4 - y^2$$

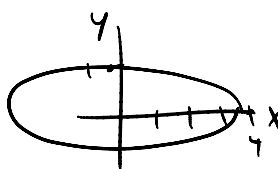
$$x^2 + 3y^2 = 4 - y^2$$

$$4y^2 = 4 - x^2$$

$$y^2 = \frac{4-x^2}{4}$$

$$y = \sqrt{1-\frac{x^2}{4}}$$

o -



$$\iint z_2 - z_1 \, dy \, dx$$

Parameterize in polar coordinate system $x = 2r \cos \theta$
 $y = r \sin \theta$

$$X = x(r, \theta) \quad dy \, dx = 2r \, dr \, d\theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr \, d\theta$$

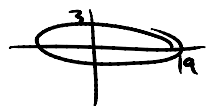
 $x = r \cos \theta$
 $y = r \sin \theta$

$$\begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta$$

determinant

$$= r$$

$$\int r \, dr \, d\theta$$

If your ellipse was  then $x = 3 \cos \theta$
 $y = 1 \sin \theta$
 & use that to
 find the
 $dr \, d\theta$

Density, Angular Momentum, Moments of Inertia

Thursday, January 11, 2007
10:03 AM

Lamina of density $\delta(x,y) = \left(\frac{\text{mass}}{\text{area}}\right)$ in shape of region R in (x,y) plane



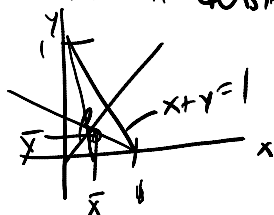
$$\bar{x} = \frac{1}{M} \iint_R \delta(x,y) x dA$$

$$\bar{y} = \frac{1}{M} \iint_R \delta(x,y) y dA$$

$$M = \text{mass of lamina} = \iint_R \delta(x,y) dA$$

Ex

Compute center of mass $(\bar{x}, \bar{y})$ of triangle w/ vertices $(0,0)$, $(1,0)$, $(0,1)$ & constant density δ_0 .

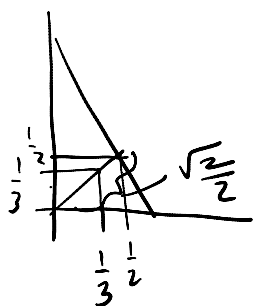


1st calculate total mass of the lamina

$$M = \delta_0 \text{ Area} = \delta_0 \left(\frac{1}{2}\right)$$

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(1)(1)$$



$$\begin{aligned} \bar{x} &= \frac{1}{M} \iint_R \delta_0 x dA = \frac{\delta_0}{\delta_0 \frac{1}{2}} \int_0^1 \int_0^{1-y} x dx dy \\ &= 2 \int_0^1 \left[\frac{x^2}{2} \right]_0^{1-y} dy = 2 \int_0^1 \frac{(1-y)^2}{2} dy \\ &= \left[-\frac{(1-y)^3}{3} \right]_0^1 = \frac{1}{3} \end{aligned}$$

Angular Momentum and Moments of Inertia

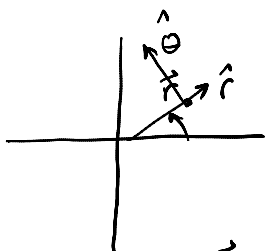
Motivating example:

Ω = angular speed

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$



I WANT TO BELIEVE



$$\vec{r}_1 = r_1 \hat{r}$$

$$\vec{r}_2 = r_2 \hat{r}$$

$$\vec{p}_1 = m_1 \vec{v}_1 = m_1 \Omega r_1 \hat{\theta}$$

$$\vec{p}_2 = m_2 \vec{v}_2 = m_2 \Omega r_2 \hat{\theta}$$

$\vec{r}_1 \times \vec{p}_1$ points out of the screen

$\vec{r}_2 \times \vec{p}_2$ points out of the screen

$$\vec{L} = \hat{k} [m_1 r_1^2 \Omega + m_2 r_2^2 \Omega]$$

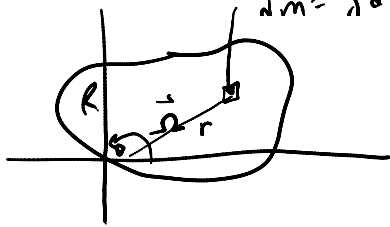
$$\vec{L} = \hat{k} [m_1 r_1^2 + m_2 r_2^2] \Omega = \pm \Omega \hat{k}$$

$I = \text{moment of inertia}$

$$\vec{L} = \pm I \vec{\Omega}$$

small mass

$$dm = \rho dx dy = \rho dA$$



$\vec{\Omega}$ out of the screen

$$I = \iint \rho r^2 dA$$

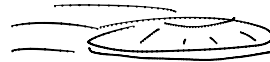
R add em all up

Moment of inertia of lamina.

$$\vec{L} = \pm \Omega \hat{k} = I \vec{\Omega}$$

$\rho(x, y) = \text{density (mass/area)}$

$r = \text{distance from rotation axis to mass elmt at } (x, y) = \sqrt{x^2 + y^2}$

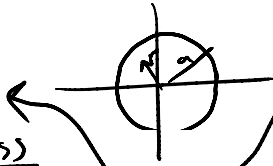


Ex 1

Calculate I of a uniform disk of mass M & radius a about its center.

$$I = \frac{1}{2} M a^2$$

$$\rho = \frac{M}{\pi a^2} \quad \frac{\text{mass}}{\text{area}}$$



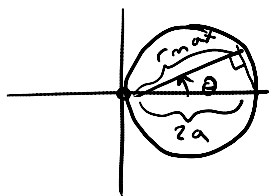
$$I = \iint \rho r^2 dA$$

$$I = \frac{M}{\pi a^2} \int_0^{2\pi} \int_0^a r^3 dr d\theta$$

b/c circular, so set up for polar

$$I = \frac{M}{\pi a^2} (2\pi) \left[\frac{r^4}{4} \right]_0^a = \frac{2M}{a^2} \left(\frac{a^4}{4} \right) = \frac{1}{2} M a^2 \quad \checkmark$$

Now we rotate it about its edge



$$r_{\max} = 2a \cos \theta \quad (\text{see yesterday's notes})$$

$$I = \iiint r^2 dA$$

$$I = \frac{M}{\pi a^2} \int_{-\pi/2}^{\pi/2} \int_0^{2a \cos \theta} r^3 dr d\theta$$

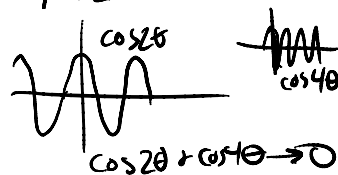
$$I = \frac{M}{\pi a^2} \int_{-\pi/2}^{\pi/2} \frac{r^4}{4} \Big|_0^{2a \cos \theta} d\theta = \frac{M}{\pi a^2} \int_{-\pi/2}^{\pi/2} 4a^4 \cos^4 \theta d\theta$$

$$= \frac{M 4a^4}{\pi a^2} \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta$$

$$(\cos^2 \theta)^2 = \frac{1}{4}(1 + \cos 2\theta)^2$$

$$= \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)$$

$$= \frac{1}{4} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta \right)$$



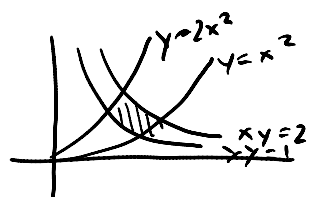
$$= \frac{3}{8} \pi \frac{M}{\pi a^2} 4a^4$$

$$= \frac{3}{2} M a^2$$

Moment of inertia is $3 \times$
 a : much when you switch
the axis.

Parallel Axis Theorem

Friday, January 12, 2007
10:03 AM

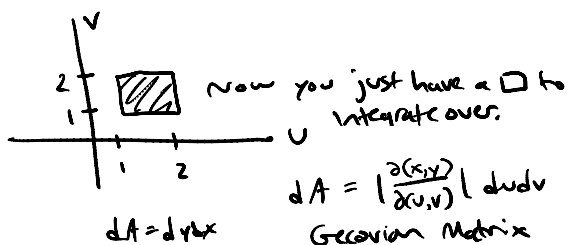


if you defined new coordinates for this

$$xy = u$$

$$\frac{y}{x} = v$$

Polar coordinates are like this bc it's changing your coordinate system so it's easier.

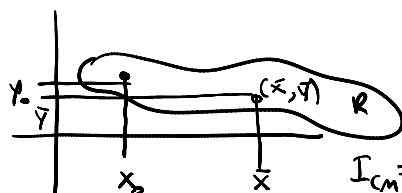


Tuesday

Midterm's on multiple integration

Parallel Axis Theorem

Let $I(x_0, y_0)$ = moment of inertia about a point (x_0, y_0) in a lamina in shape of R w/ density $\delta(x, y)$



This can be related to center of mass of object. Let $(\bar{x}, \bar{y})$ = center of mass

$I_{cm} = I(\bar{x}, \bar{y})$ = moment of inertia about center of mass

Let $d = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2}$ distance from $(\bar{x}, \bar{y})$ to (x_0, y_0)

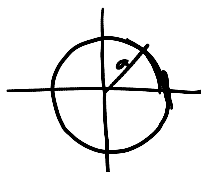
M = mass of lamina.

$$I(x_0, y_0) = I_{cm} + M d^2$$

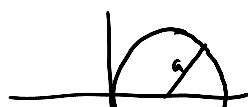
Corollary

The minimum value of $I(x_0, y_0)$ occurs at $(x_0, y_0) = (\bar{x}, \bar{y})$ and is the moment of inertia about center of mass, I_{cm} . $d = 0$.

Yesterday's example

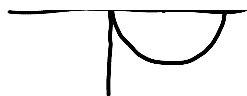


$$I = \frac{1}{2} M a^2$$



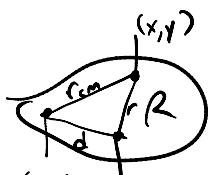
$$I = I_{cm} + M d^2 ; d = a$$

$$I = \frac{1}{2} M a^2 + M a^2$$



$$I = \frac{3}{2} M a^2$$

So this is way faster than yesterday.



$\delta(x, y)$ = density
Region R

$(\bar{x}, \bar{y})$ $(x_0, y_0) \rightarrow$ doesn't have to be in R for it to apply

(x, y) = coordinates of an arbitrary point in R

$(\bar{x}, \bar{y})$ = center of mass

(x_0, y_0) = position of axis. (calculate I about this pt)

$$I(x_0, y_0) = \iint_R \delta r^2 dA = ? \quad \iint_R \delta r_{cm}^2 dA + M d^2$$

$$r_{cm}^2 = (x - \bar{x})^2 + (y - \bar{y})^2$$

$$= \iint_R \delta [(x - x_0)^2 + (y - y_0)^2] dA$$

$$(x - x_0)^2 = (x - \bar{x} + \bar{x} - x_0)^2 = (x - \bar{x})^2 + (\bar{x} - x_0)^2 + 2(x - \bar{x})(\bar{x} - x_0)$$

$$(y - y_0)^2 = (y - \bar{y} + \bar{y} - y_0)^2 = (y - \bar{y})^2 + (\bar{y} - y_0)^2 + 2(y - \bar{y})(\bar{y} - y_0)$$

$$\underbrace{(x - \bar{x})^2 + (y - \bar{y})^2}_{r_{cm}^2} + \underbrace{(\bar{x} - x_0)^2 + (\bar{y} - y_0)^2}_{d^2} + 2[(x - \bar{x})(\bar{x} - x_0) + (y - \bar{y})(\bar{y} - y_0)]$$

$$I(x_0, y_0) = \underbrace{\iint_R \delta r_{cm}^2 dA}_{I_{cm}} + \underbrace{d^2 \iint_R \delta dA}_M + 2(\bar{x} - x_0) \underbrace{\iint_R \delta x dA}_{M\bar{x}} - \bar{x} \underbrace{\iint_R \delta dA}_M$$

$$\bar{x} = \frac{1}{M} \iint_R \delta x dA$$

$$+ 2(\bar{y} - y_0) \left[\underbrace{\iint_R \delta y dA}_{M\bar{y}} - \bar{y} \underbrace{\iint_R \delta dA}_M \right]$$

$$\text{and so } M\bar{x} - \bar{x}M = 0$$

$$M\bar{y} - \bar{y}M = 0$$

$$\text{so } I(x_0, y_0) = \underbrace{\iint_R \delta r_{cm}^2 dA}_{I_{cm}} + d^2 \underbrace{\iint_R \delta dA}_M$$

Change of variables for Double Integrals

1D

$$\int f(x) dx$$

Change from x-variable to u-variable

Let $x = x(u) \Rightarrow dx = \frac{dx}{du} du$ $\frac{\text{change in } x}{\text{change in } u} = \text{ratio of lengths}$

$$\int f(x) dx = \int f(x(u)) \frac{dx}{du} du$$

Usually you do it like this:

Method of substitution

$$\int f(x) dx$$

Let $u = u(x)$

$$du = \frac{du}{dx} dx$$

$$\Rightarrow dx = \left(\frac{1}{\frac{du}{dx}} \right) du$$

In the end you need $\int f(u) du$

2D

$$\iint_R f(x,y) \frac{dx}{dy}$$

old variables (x,y)

new variables (u,v)

Here we can take a complete region & make it into a nice region.

$$x = X(u,v)$$

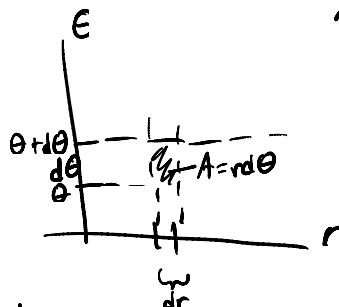
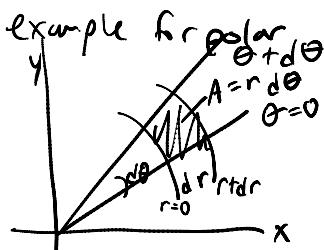
$$y = Y(u,v)$$

Ex $u=r$ $x=r \cos \theta$

$v=\theta$ $y=r \sin \theta$

$$\iint_R f(X(u,v), Y(u,v)) dA$$

? We want to find the ratio of area elmts. $= \frac{\text{Area in } (x,y)}{\text{Area in } (u,v)} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$



$$dA = r dr d\theta$$

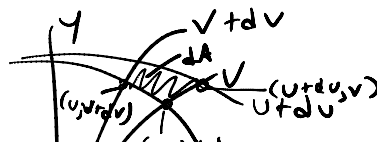
$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|$$

General coordinate transformation

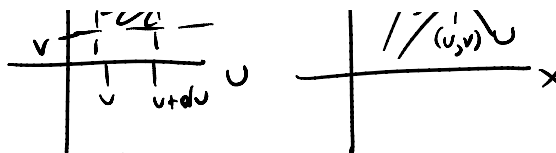
$$(x,y) \rightarrow (u,v)$$

$$x = X(u,v)$$

$$v+dv$$



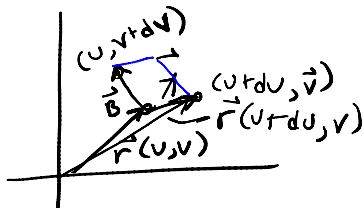
$$\vec{r} = \vec{r}(u, v)$$



These lines transfer to some curve in x, y plane. Hold u constant & vary v .
Now increase u in amt du

Then draw one that holds v fixed

Use a $\square$ to estimate area elmt dA



Area is mag cross product of 2 vectors. $dA = |\vec{A} \times \vec{B}|$

$$\begin{aligned} \vec{A} &= \vec{r}(u+du, v) - \vec{r}(u, v) \\ &= \langle X(u+du, v), Y(u+du, v) \rangle - \langle X(u, v), Y(u, v) \rangle \end{aligned}$$

Taylor expand X and Y about $du=0$

Linear approx

$$f(x) \text{ about } h = f(x) + \frac{df}{dx}(x) h + f''(x) \frac{h^2}{2}$$

$$\approx \langle X(u, v) + \frac{\partial X}{\partial u} du, Y(u, v) + \frac{\partial Y}{\partial u} du \rangle - \langle X(u, v), Y(u, v) \rangle$$

$$\vec{A} = \left[\vec{r}(u, v) + \frac{\partial \vec{r}}{\partial u}(u, v) du \right] - \vec{r}(u, v) = \frac{\partial \vec{r}}{\partial u} du = \left\langle \frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u} \right\rangle du$$

$$\vec{B} = \left\langle \frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v} \right\rangle dv = \frac{\partial \vec{r}}{\partial v} dv$$

$$dA = |\vec{A} \times \vec{B}| = \left| \frac{\partial \vec{r}}{\partial u} du \times \frac{\partial \vec{r}}{\partial v} dv \right| = \underbrace{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}_{\left| \frac{\partial(x, y)}{\partial(u, v)} \right| \leftarrow \text{Jacobian}} du dv$$

Jacobian

$$\vec{r} = \langle X(u, v), Y(u, v), 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, 0 \right\rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, 0 \right\rangle$$

polar coordinates

$$\text{ex. } \vec{r} = \langle X(r, \theta), Y(r, \theta) \rangle = \langle r \cos \theta, r \sin \theta \rangle$$

$$u=r$$

$$v=\theta$$

$$\frac{\partial \vec{r}}{\partial r} = \left\langle \frac{\partial X}{\partial r}, \frac{\partial Y}{\partial r}, 0 \right\rangle = \langle \cos \theta, \sin \theta, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} = \left\langle \frac{\partial X}{\partial \theta}, \frac{\partial Y}{\partial \theta}, 0 \right\rangle = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & 0 \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & 0 \end{vmatrix}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \hat{k} \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

$$\text{or } d\vec{r} = \begin{vmatrix} -r \sin \theta & r \cos \theta \\ -r \cos \theta & -r \sin \theta \end{vmatrix} = r \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= r \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right)$$

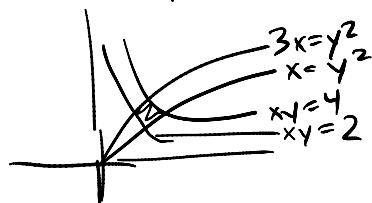
$$\iint_R F(x, y) dx dy = \iint_R F(x(u, v), y(u, v)) \underbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}_{dA} du dv$$

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = x(u, v)$
 $y = y(u, v)$

Ex Find

$$\iint_R x^4 y \, dx dy$$

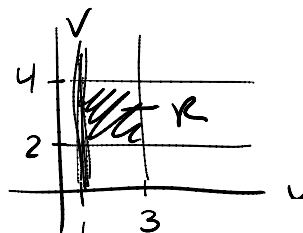


So region is complicated

$$\text{Let } u = y^2/x$$

$$v = xy$$

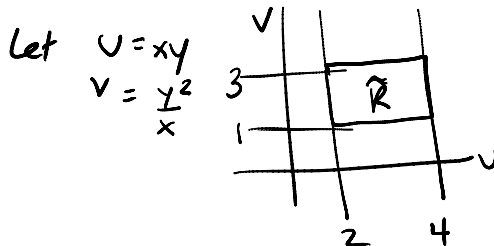
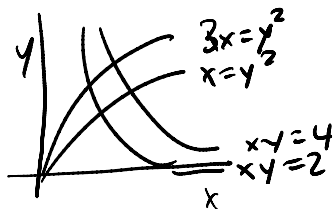
$$\int_2^4 \int_1^3 \boxed{} du dv$$



Ex., cylindrical, Spherical coordinates

Wednesday, January 17, 2007
10:03 AM

ex. $\iint_R x^4 y \, dx \, dy$



$$\int_1^3 \int_2^4 (x(u,v))^4 (y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \, dv$$

$$u \cdot v = y^3$$

$$v = u^{\frac{1}{3}} v^{\frac{1}{3}}$$

$$x = \frac{u}{y} = \frac{u}{u^{\frac{1}{3}} v^{\frac{1}{3}}} = u^{\frac{2}{3}} v^{-\frac{1}{3}}$$

$$x^4 y = \left(u^{\frac{8}{3}} v^{-\frac{4}{3}} \right) u^{\frac{1}{3}} v^{\frac{1}{3}} = u^{\frac{9}{3}} v^{-\frac{3}{3}} = \frac{u^3}{v}$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} - \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix}$$

$$= \left(\frac{2}{3} u^{-\frac{1}{3}} v^{-\frac{1}{3}} \right) \left(\frac{1}{3} u^{\frac{1}{3}} v^{-\frac{2}{3}} \right) - \left(-\frac{1}{3} u^{\frac{2}{3}} v^{-\frac{4}{3}} \right) \left(\frac{1}{3} u^{-\frac{2}{3}} v^{\frac{1}{3}} \right)$$

$$\int_1^3 \int_2^4 \frac{u^3}{v} \left(\frac{1}{3v} \right) du \, dv = \frac{1}{3} v^{-1} = \frac{1}{3v}$$

Area must be pos, so you could need 1 sign, but you don't b/c x & y are (+) ranges (1-3, 2-4)

$$\frac{1}{3} \int_1^3 \frac{1}{v^2} \left[\int_2^4 u^3 du \right] dv = \frac{1}{3} \left[\int_1^3 \frac{1}{v^2} dv \right] \left[\int_2^4 u^3 du \right]$$

This works b/c region is rectang

$$= \frac{1}{3} \left(-\frac{1}{v} \right) \Big|_1^3 \cdot \left(\frac{u^4}{4} \right) \Big|_2^4$$

$$= \frac{1}{3} \left(\frac{2}{3} \right) \cdot (64 - 4) = \frac{40}{3}$$

Observe

$$du \, dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx \, dy$$

$$dxdy = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx dy$$

$$dxdy = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} du dv$$

$$\therefore dxdy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

$$\text{Therefore } \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \right|$$

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \left| \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right|$$

$$= \left| y \left(\frac{2y}{x} - x \left(-\frac{y^2}{x^2} \right) \right) \right| = \left| \frac{2y^2}{x} + \frac{y^2}{x} \right| = \frac{3y^2}{x}$$

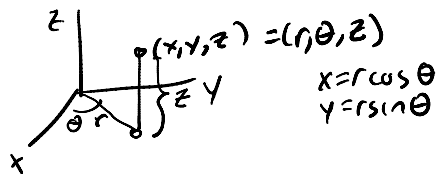
$$\int_1^3 \int_2^4 x^4 y \left(\frac{x}{3y^2} \right) du dv$$

$$= \frac{1}{3} \int_1^3 \int_2^4 \frac{x^5}{y} du dv$$

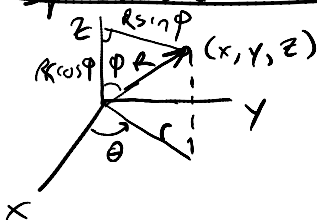
$$\downarrow$$

$$\frac{1}{3} \int_1^3 \int_2^4 \frac{x^5}{y^2} du dv = \frac{x^3}{y^4} x^2 = \frac{x^5}{y} \checkmark$$

Cylindrical Coordinates



Spherical coordinates



$$R^2 = x^2 + y^2 + z^2$$

$$z = R \cos \phi$$

$$x = R \sin \phi \cos \theta$$

$$y = R \sin \phi \sin \theta$$

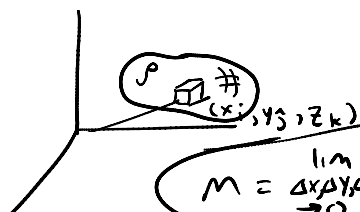
$$\frac{y}{x} = \tan \theta$$

$$\frac{\sqrt{x^2 + y^2}}{z} = \frac{R \sin \phi}{R \cos \phi}$$

$$\frac{\sqrt{x^2 + y^2}}{z} = \tan \phi$$

Triple Integrals

We want to calculate the mass of some volume V , with density $\rho(x, y, z)$

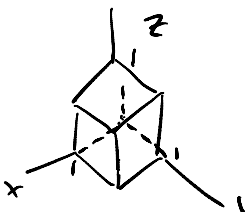


$$m = \rho(x_i, y_j, z_k) \Delta x \Delta y \Delta z$$

$$m = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \left\{ \sum_{x_i, y_j, z_k \in V} \rho(x_i, y_j, z_k) \Delta x \Delta y \Delta z \right\}$$

$$m = \int \int \int_V \rho(x, y, z) dx dy dz$$

ex.



$$\int_0^1 \int_0^1 \delta(x, y) dx dy$$

$$\int_0^1 \int_0^1 \left(\int_0^1 \rho(x, y, z) dz \right) dy dx$$

area integral & then 2 outside surfaces

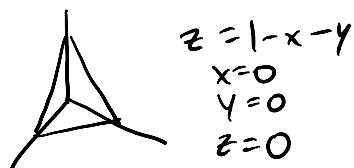
ex2

$$\begin{aligned}
 \rho(x, y, z) &= x^2 + y^2 + z^2 \\
 \int_0^1 \int_0^1 \left(\int_0^1 (x^2 + y^2 + z^2) dz \right) dy dx \\
 \int_0^1 \int_0^1 \left((x^2 + y^2)z + \frac{z^3}{3} \right) \Big|_0^1 dy dx \\
 \int_0^1 \int_0^1 \left((x^2 + y^2) + \frac{1}{3} \right) dy dx \\
 \int_0^1 \left[xy + \frac{y^3}{3} + \frac{1}{3}y \right]_0^1 dx \\
 \int_0^1 \left(x^2 + \frac{2}{3} \right) dx \\
 \left[\frac{x^3}{3} + \frac{2}{3}x \right]_0^1 &= 1
 \end{aligned}$$

Now how do we decide which to do 1st? Find which

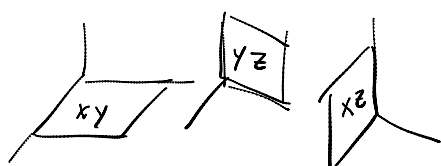
z are the easiest to project the volume on?

ex₃ The Triforce



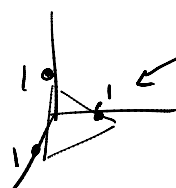
Setting up triple integrals

① Sketch each surface separately.



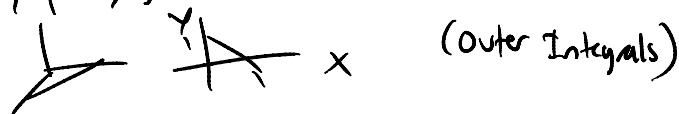
$x=0$, yz plane
 $y=0$, xz plane
 $z=0$, xy plane

$z = 1 - x - y$, Δ plane at angle.



② Choose 2 variables whose plane is the easiest to integrate after taking the volume's projection.

Collapse into xy plane, get Δ . We know how to integrate a Δ

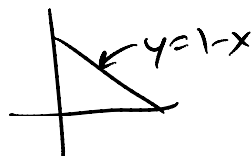


③ The 3rd variable corresponds to the inner integral whose limits are the upper & lower surfaces

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y}$$

$$dz \, dy \, dx$$

outers are xy b/c we chose xy plane.



$$V = \int_0^1 \int_0^{1-x} z \, dy \, dx$$

$$V = \int_0^1 \int_0^{1-x} (1-x-y) \, dy \, dx$$

$$V = \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$V = \int_0^1 \left((1-x) - x(1-x) - \frac{(1-x)^2}{2} \right) dx$$

$$V = \int_0^1 -\frac{1}{6} (1-x)^3 dx$$

$$V = \frac{1}{6}$$

ex₄ The Spring Doorstop of Doom

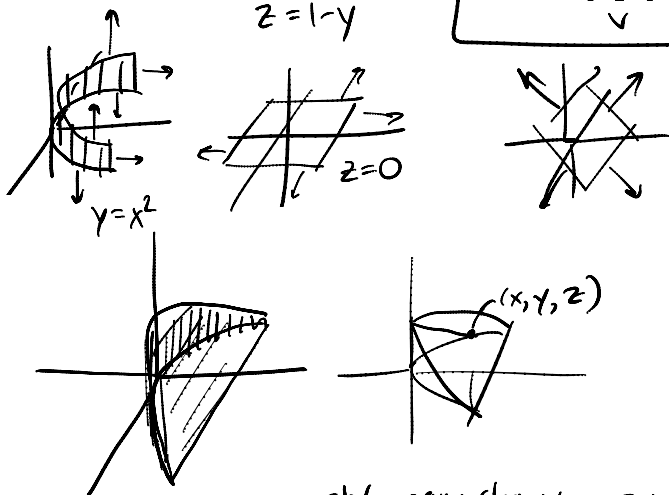
Calculate the moment of inertia of the volume V about the z axis. $V = x^2$

$$\rho = 1$$

$$\begin{aligned} z &= 0 \\ z &= 1-y \end{aligned}$$

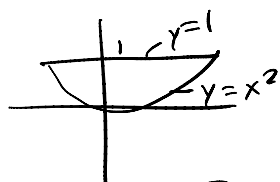
$$I = \iiint_V \rho \, d^3V$$

①



②

$$d^2 = x^2 + y^2 + dz^2 = x^2 + y^2$$



outer integrals

upper + lower surfaces, $z = 1-y$

$$I = \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} (x^2 + y^2) \, dz \, dy \, dx$$

$$I = \int_{-1}^1 \int_{x^2}^1 (x^2 + y^2) z \Big|_0^{1-y} \, dy \, dx$$

$$I = \int_{-1}^1 \int_{x^2}^1 (x^2 + y^2) (1-y) \, dy \, dx$$

$$I = \int_{-1}^1 \int_{x^2}^1 (x^2 - yx^2 + y^2 - y^3) \, dy \, dx$$

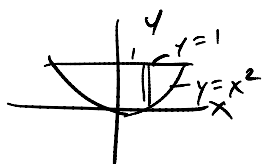
$$I = \int_{-1}^1 \left[yx^2 - \frac{y^2}{2}x^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_{x^2}^1 \, dx$$

$$I = \int_{-1}^1 \left(x^2 - \frac{x^2}{2} + \frac{1}{3} - \frac{1}{4} \right) - \left(x^4 - \frac{x^6}{2} + \frac{x^6}{3} - \frac{x^8}{4} \right) \, dx$$

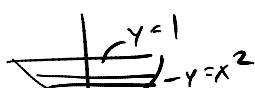
$$I = \int_{-1}^1 \left(\frac{1}{2} + \frac{1}{2}x^2 - x^4 + \frac{1}{6}x^6 + \frac{1}{4}x^8 \right) \, dx$$

$$I = \left[\frac{1}{12}x + \frac{x^3}{6} - \frac{x^5}{5} + \frac{x^7}{42} + \frac{x^9}{36} \right]_{-1}^1$$

$$I = \text{some \#} \quad \frac{64}{315}$$

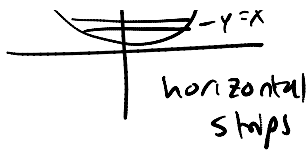


$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} (x^2 + y^2) \, dz \, dy \, dx$$



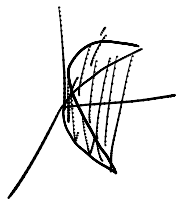
$$\int_{-1}^1 \int_{x^2}^1 (x^2 + y^2) \, dz \, dx \, dy$$

Just changes

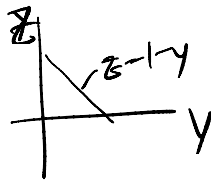


$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} (x^2 + y^2) dz dx dy$$

outer z .



$$\begin{aligned} z &= 1-y \\ x^2 &= y \\ z &= 0 \end{aligned}$$



vertical strips

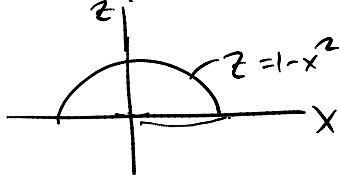
$$\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} (x^2 + y^2) dx dz dy$$

b/c looking from x direction in, lower surface is parabola.

horizontal strips

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} (x^2 + y^2) dx dy dz$$

xz plane

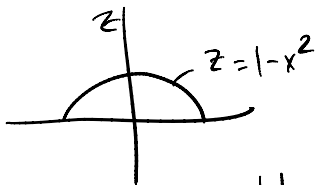


vertical strips

$$\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} (x^2 + y^2) dy dz dx$$

$1-z \leftarrow y = 1-z$, the Δ .

x^2
↑
parabola



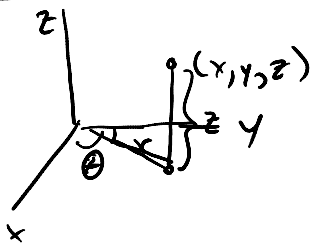
horizontal strips

$$\int_0^1 \int_{\sqrt{1-z}}^{\sqrt{1-z}} \int_{-x^2}^{1-z} (x^2 + y^2) dy dx dz$$

Cylindrical Coordinates

Monday, January 22, 2007
10:02 AM

Cylindrical Coordinates

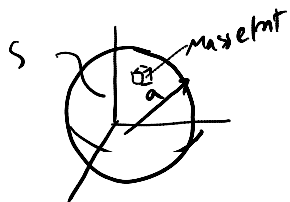


Volume element

$$dV = \underbrace{r dr d\theta dz}_{dx dy}$$

ex 1

Compute the moment of inertia about the z axis of a sphere of radius a , constant density, and total mass M . ^{centered at origin}



$$I = \iiint_S \rho d^2 dV \leftarrow r dr d\theta dz$$

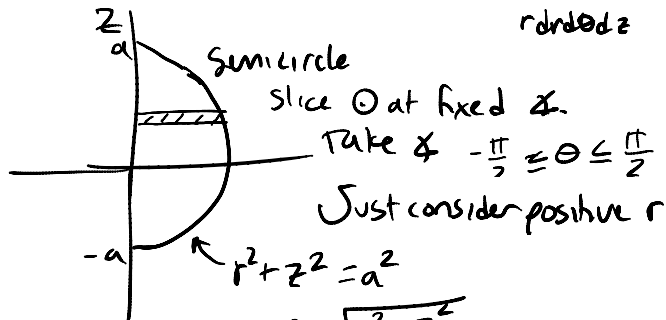
density

$d = \text{distance from } (x, y, z) = \sqrt{x^2 + y^2} = r$

$$\rho_0 = \text{constant density} = \frac{\text{total mass } M}{\text{volume } (\frac{4}{3}\pi a^3)} = \frac{3M}{4\pi a^3}$$

In cylindrical coordinates, $r^2 = x^2 + y^2$ so this is the same

$$I = \frac{3M}{4\pi a^3} \iiint_S r^2 dV = \frac{3M}{4\pi a^3} \int_{-a}^a \int_0^{2\pi} \int_0^{\sqrt{a^2 - z^2}} r^3 dr d\theta dz$$



$$I = \int_{-a}^a \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^{\sqrt{a^2 - z^2}} d\theta dz$$

2π b/c it's symmetric. So you know you'll get a factor of 2π .

$$I = \frac{3M}{4\pi a^3} (2\pi) \frac{1}{4} \int_{-a}^a (a^2 - z^2)^2 dz = \frac{3M}{8a^3} \int_{-a}^a (a^4 - 2a^2 z^2 + z^4) dz$$

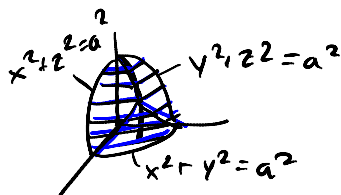
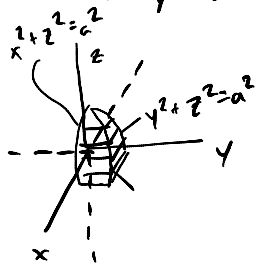
$$= \frac{3M}{8a^3} \left(a^4 z - \frac{2}{3} a^2 z^3 + \frac{1}{5} z^5 \right) \Big|_{-a}^a$$

$$= \frac{3M}{8a^3} (2a^5) \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{8}{15} M a^2$$

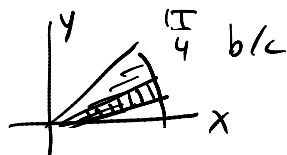
ex 2

Volume common to 3 cylinders

$$x^2 + y^2 = a^2 ; x^2 + z^2 = a^2 ; y^2 + z^2 = a^2$$



Calculate $\frac{1}{6}$ intersection
in cylindrical coordinates



$$V = 16 \int \int \int_{\frac{1}{16} T} dV = 16 \int \int \int_{\frac{1}{16} T} r dr d\theta dz$$

in order $dz dr d\theta$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^a \int_0^{\sqrt{a^2 - x^2}} r dz dr d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^a \int_0^{\sqrt{a^2 - r^2 \cos^2 \theta}} r dz dr d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \left[\frac{1}{2} r^2 \right]_0^{\sqrt{a^2 - r^2 \cos^2 \theta}} d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^a r \sqrt{a^2 - r^2 \cos^2 \theta} dr d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^a \frac{1}{2} \frac{du}{-2r \cos^2 \theta} \sqrt{u} du d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \left[\frac{1}{3} \frac{u^{3/2}}{-2 \cos^2 \theta} \right]_0^a d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{4}} \left(a^2 \left(\frac{1}{2} \right) \left(\frac{1}{-\cos^2 \theta} \right) \right) d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{a^3}{\cos^2 \theta} - \frac{(a^3 ((-\cos^2 \theta))^{\frac{3}{2}})}{\cos^2 \theta} d\theta$$

$$= \frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{a^3}{\cos^2 \theta} - \frac{a^3 \sin^3 \theta}{\cos^2 \theta} d\theta$$

$$= \frac{16}{3} a^3 \int_0^{\frac{\pi}{4}} \underbrace{\sec^2 \theta}_{\tan \theta} - \frac{\sin^3 \theta}{\cos^2 \theta} d\theta \quad \leftarrow \frac{\sin \theta \sin^2 \theta}{\cos^2 \theta} \frac{\sin \theta \cos \theta}{\cos^2 \theta}$$

$$= \frac{16}{3} a^3 (1) - \int_0^{\frac{\pi}{4}} \underbrace{\frac{\sin \theta}{\cos^2 \theta}}_{\frac{1}{\cos \theta} + \cos \theta} - \sin \theta d\theta$$

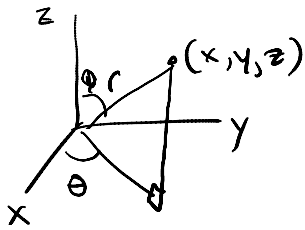
$$= \frac{16a^3}{3} - ((\sqrt{2} - 1) + (\frac{1}{\sqrt{2}} - 1))$$

dr ends up
with
 $x^2 + y^2 = a^2$,
 $a^2 = r^2$,
 $r = a$

$$= \frac{16a^3}{3} - \left((\sqrt{2} - 1) + \left(\frac{1}{\sqrt{2}} - 1 \right) \right)$$

$$= \frac{16a^3}{3} 3 \left(1 - \frac{\sqrt{2}}{2} \right)$$

Spherical Coordinates

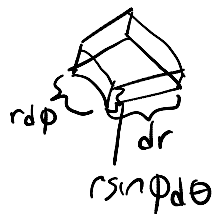
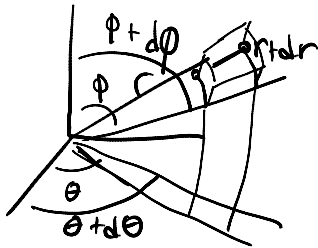


$$x = r \sin \phi \cos \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \phi$$

$$dV = r^2 \sin \phi \, dr \, d\phi \, d\theta$$



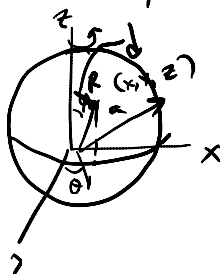
product of 3 is
like a cube = volume

Triple integrals in Spherical coordinates

ex 1

Compute the moment of inertia of a sphere about an axis through its center

Radius a
Density ρ_0 , constant = , mass M



$$I = \iiint_S \rho d^2 dV$$

d = distance from (x, y, z) to z axis.

$$R^2 \sin \phi \, dR \, d\phi \, d\theta$$

$$d = \sqrt{x^2 + y^2} \quad \text{because same } z \text{ value}$$

$$= R \sin \phi$$

$$\rho = \frac{M}{\frac{4}{3}\pi a^3} = \frac{3M}{4\pi a^3}$$

$$I = \frac{3M}{4\pi a^3} \int_0^{2\pi} \int_0^\pi \int_0^a R^4 \sin^3 \phi \, dR \, d\phi \, d\theta$$

B/c it's a sphere, none of integrals depend on variables.

The bounds are independent-

$$\int_0^a R^4 \sin^3 \phi \, dR = \frac{R^5}{5} \Big|_0^a$$

$$\frac{3M}{4\pi a^3} \left(\frac{a^5}{5} \right) (2\pi) \int_0^\pi \sin^3 \phi \, d\phi$$

$$\frac{3M}{10} a^2 \int_0^\pi \sin \phi (1 - \cos^2 \phi) \, d\phi$$

$$\int_0^\pi \sin \phi - \sin \phi \cos^2 \phi \, d\phi$$

$$\left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^\pi \quad \begin{matrix} u = \cos \phi \\ du = -\sin \phi \, d\phi \end{matrix}$$

$$\left((1) - \frac{1}{3} \right) - \left((-1) + \frac{1}{3} \right)$$

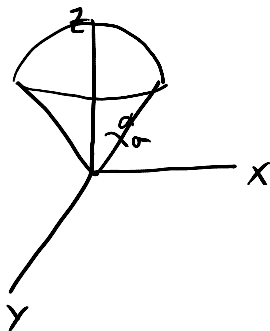
$$\left(\frac{2}{3} \right) - \left(-\frac{2}{3} \right) = \frac{4}{3}$$

$$\frac{3}{10} M a^2 \left(\frac{4}{3} \right) = \boxed{\frac{2}{5} M a^2}$$

This is what we got from cylindrical coordinates.

One typically uses spherical coordinates when one has a sphere.
Cylindrical coordinates are for when you have a symmetric thing.

Cylindrical coordinates are for when you have a symmetric thing
ex. 2 Use spherical coordinates to compute the center of mass of
 the "spherical cone" radius = a



angle to z axis = α

constant density ρ_0

$$M = \rho_0 V = \rho_0 \iiint_S dV$$

$$\bar{x} = \frac{1}{M} \iiint_S \rho_0 x dV = 0$$

$$\bar{y} = \frac{1}{M} \iiint_S \rho_0 y dV = 0$$

} symmetry
 b/c of
 uniform
 density

$$\bar{z} = \frac{1}{M} \iiint_S \rho_0 z dV$$

$$dV = R^2 \sin \phi dR d\phi d\theta$$

$$\rho_0 \iiint_0^{2\pi} \int_0^\alpha \int_0^a R^2 \sin \phi dR d\phi d\theta$$

Again, bounds don't depend on
 variables

$$\rho_0 (2\pi) (\cos \phi)_0^\alpha \left(\frac{R^3}{3} \right)_0^a$$

$$\rho_0 - 2\pi (\cos \alpha - 1) \left(\frac{a^3}{3} \right)$$

$$\rho_0 \left(2\pi \frac{a^3}{3} (1 - \cos \alpha) \right) = M$$

You can check the limits here. if $\alpha = 0$,
 then you get

Taylor series for cos

$$\cos \alpha = 1 - \frac{\alpha^2}{2}$$

$$\left(\frac{4}{3} \pi a^3 \right) \left(\frac{\alpha^2}{4} \right)$$

Volume

fraction

$$\bar{z} = \frac{1}{M} \iiint_S \rho_0 z dV$$

$$\bar{z} = \frac{3}{2\pi a^3 \rho_0 (1 - \cos \alpha)} \int_0^{2\pi} \int_0^\alpha \int_0^a z dV$$

$$\bar{z} = \frac{3}{2\pi a^3 (1 - \cos \alpha)} \int_0^{2\pi} \int_0^\alpha \int_0^a R^3 \sin \phi \cos \phi dR d\phi d\theta$$

$$(2\pi) \left(\frac{\sin^2 \phi}{2} \right)_0^\alpha \left(\frac{a^4}{4} \right)$$

$$\left(\frac{1 - \cos^2 \phi}{2} \right)_0^\alpha$$

$$\frac{3a}{8} \frac{\sin^2 \alpha}{1 - \cos \alpha} = \frac{3a}{8} \frac{(1 - \cos^2 \alpha)}{1 - \cos \alpha} = \frac{3a}{2} (1 + \cos \alpha) \geq 0$$

$$\frac{\frac{3a}{8} \frac{\sin^2 \alpha}{1 - \cos \alpha}}{\frac{3a}{8} \frac{(1 - \cos^2 \alpha)}{1 - \cos \alpha}} = \frac{3a}{8} (1 + \cos \alpha) \geq 0$$

as $\alpha \rightarrow \pi$, center of mass $\rightarrow 0$
 as $\alpha \rightarrow 0$,

$$0 \leq \bar{z} < \frac{3a}{4}$$



Jacobians in 3D

$$\begin{vmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \end{vmatrix} \left| \frac{\partial(R, \phi, \theta)}{\partial(x, y, z)} \right|$$

$$dR d\phi d\theta = \left| \frac{\partial(R, \phi, \theta)}{\partial(x, y, z)} \right| dx dy dz$$

$$dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(R, \phi, \theta)} \right| dR d\phi d\theta$$

$$\begin{aligned} x &= R \sin \phi \cos \theta \\ y &= R \sin \phi \sin \theta \\ z &= R \cos \phi \end{aligned}$$

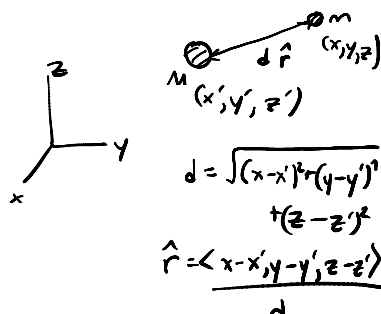
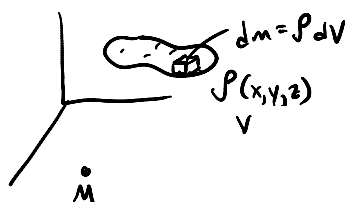
$$\begin{aligned} x^2 + y^2 + z^2 &= R^2 \\ \tan \phi &= \frac{x^2 + y^2}{z^2} \\ \tan \theta &= \frac{y}{x} \end{aligned}$$

$$\begin{aligned} \sec^2 \theta d\theta &= -\frac{y}{x^2} dx & \sec^2 \theta d\theta &= \frac{1}{x} dy \\ \frac{d\theta}{dx} &= -\frac{y}{x^2 \sec^2 \theta} & \frac{d\theta}{dy} &= \frac{1}{x \sec^2 \theta} \\ \frac{d\theta}{dz} &= 0 \end{aligned}$$

Applicational - Gravitational Attraction

Force $\vec{F}$ on a mass M due to a mass m a distance d away is

$$\vec{F} = G \frac{Mm}{d^2} \hat{r}$$

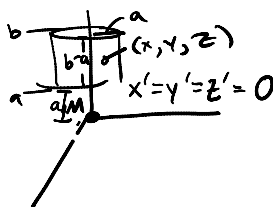


Force on M due to blob V of density $\rho(x, y, z)$ is

$$\vec{F} = GM \iiint_V \frac{\rho(x, y, z)}{d^2} \hat{r} dV$$

Ex

Gravitational force on mass M at origin due to a cylinder $0 \leq z \leq b$, $r \leq a$ in cylindrical coordinates & density constant ρ_0 .



No component of force in $-y$ direction. B/c of symmetry.
 $d = \sqrt{x^2 + y^2 + z^2}$, $\hat{r} = \frac{\langle x, y, z \rangle}{d}$

$$\vec{F} = \langle F_x, F_y, F_z \rangle$$

$$F_x = GM\rho_0 \iiint \frac{x}{d^3} dV$$

$$F_z = GM\rho_0 \iiint \frac{z}{d^3} dV$$

Cylindrical coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$dV = r dr d\theta dz$$

$$F_x = GM\rho_0 \int_0^b \int_0^{2\pi} \int_0^a \frac{r \cos \theta}{(r^2 + z^2)^{3/2}} r dr d\theta dz$$

$$\left[\frac{r^2}{(r^2 + z^2)^{3/2}} \sin \theta \right]_0^{2\pi} = 0$$

$$x^2 + y^2 + z^2 = r^2 + z^2$$

so you're done. No x component of the force.
And y is 0

$$F_z = GM\rho_0 \int_0^{2\pi} \int_0^a \int_0^b \frac{z}{(r^2 + z^2)^{3/2}} r dz dr d\theta$$

$$F_z = GM\rho_0 2\pi \int_0^a \int_0^b \frac{z}{(r^2 + z^2)^{3/2}} dz dr$$

$$= GM\rho_0 2\pi \int_0^a r \int_0^b \frac{z}{(r^2 + z^2)^{3/2}} dz dr$$

$$u = r^2 + z^2$$

$$du = 2z dz$$

$$\int \frac{1}{2} u^{-1/2} du = \frac{1}{2} u^{-1/2} (-2) \Big|_{u=r^2+a^2}^{u=r^2+b^2}$$

$$F_z = 2\pi G M \rho_0 \int_0^a \left(\frac{r}{\sqrt{r^2 + A^2}} - \frac{r}{\sqrt{r^2 + b^2}} \right) dr$$

$$\sqrt{r^2 + A^2} - \sqrt{r^2 + b^2} \Big|_0^a$$

$$\vec{F} = \vec{F}_z \hat{k} = 2\pi G M \rho_0 \left[(b-A) + \sqrt{a^2 + A^2} - \sqrt{a^2 + b^2} \right]$$

Challenge: Let $B=A+h$ assume $\frac{h}{A}$ & $\frac{a}{A}$ are $\ll 1$ ^{much less than}
 any cylinder

Show $F_z \approx \frac{GM\rho_0}{A^2} (\pi a^2 h)$ at leading order in $\frac{h}{A}$ & $\frac{a}{A}$

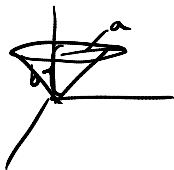
Rewrite answer

$$\vec{F} = \vec{F}_z \hat{k} = 2\pi G M \rho_0 \left[(B-A) + \sqrt{\quad} \right]$$

Taylor expand terms

2b)

Spherical



$$\vec{F} = G M \rho_0 \int_0^{2\pi} \int_0^\pi \int_0^a \frac{\vec{r}}{d^3} dV$$

$\tan \theta = \frac{a}{b} \Rightarrow \theta \approx \frac{a}{b}$ θ goes from 0 to $\frac{a}{b}$
 $d = \sqrt{r^2 + z^2}$
 $dV = r dr d\theta dz$

$$\frac{\langle x, y, z \rangle}{d^3}$$

Jacobians, Surface Integrals

Friday, January 26, 2007

10:03 AM

Jacobians for triple integrals

$$\iiint_T f(x, y, z) dx dy dz = \iiint f(x(u, v, w), y(u, v, w), z(u, v, w))$$

$$x = x(u, v, w)$$

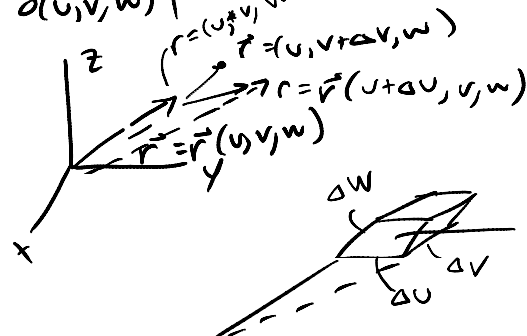
$$y = y(u, v, w)$$

$$z = z(u, v, w)$$

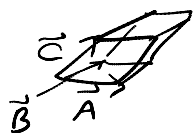
$$(x, y, z) \rightarrow (u, v, w)$$

$$\text{eg } (x, y, z) = (R, \phi, \sigma)$$

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$



Parallel piped w/ volume ΔV



$$V = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

Triple product.



$A \times B = \text{Area of base.}$

$$\Delta V = |(\vec{A} \times \vec{B}) \cdot \vec{C}|$$

$$\Delta V = |(\vec{A} \times \vec{C}) \cdot \vec{B}|$$

$$\Delta V = |(\vec{B} \times \vec{C}) \cdot \vec{A}|$$

All the same

Parentheses not needed

b/c $\vec{B} \cdot \vec{C} = \text{scalar}$,

$\vec{A} \times \text{scalar} = ? \text{ nothing.}$

$$\vec{A} = \vec{r}(u+\Delta u, v, w) - \vec{r}(u, v, w)$$

taylor expand about $\Delta u = 0$

$$\vec{A} \approx \left[\vec{r}(u, v, w) + \frac{\partial \vec{r}}{\partial u}(u, v, w) \Delta u \right] - \vec{r}(u, v, w)$$

$$\vec{A} \approx \frac{\partial \vec{r}}{\partial u} \Delta u$$

$$\vec{r} = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle$$

$$\vec{A} \approx \frac{\partial \vec{r}}{\partial u} \Delta u$$

$$\vec{B} \approx \frac{\partial \vec{r}}{\partial v} \Delta v$$

$$\vec{C} \approx \frac{\partial \vec{r}}{\partial w} \Delta w$$

$$V \approx \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \cdot \frac{\partial \vec{r}}{\partial w} \Delta u \Delta v \Delta w = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u \Delta v \Delta w$$

$$V \approx \left(\frac{\partial \vec{r}}{\partial u} \Delta u \times \frac{\partial \vec{r}}{\partial v} \Delta v \right) \cdot \frac{\partial \vec{r}}{\partial w} \Delta w = \left| \frac{\partial \vec{r}}{\partial u} \Delta u \cdot \left(\frac{\partial \vec{r}}{\partial v} \Delta v \times \frac{\partial \vec{r}}{\partial w} \Delta w \right) \right|$$

$$V \approx \left| \frac{\partial \vec{r}}{\partial u} \cdot \left(\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial w} \right) \right| \Delta u \Delta v \Delta w$$

$$dV = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

$$\frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$\frac{\partial \vec{r}}{\partial w} = \left\langle \frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w} \right\rangle$$

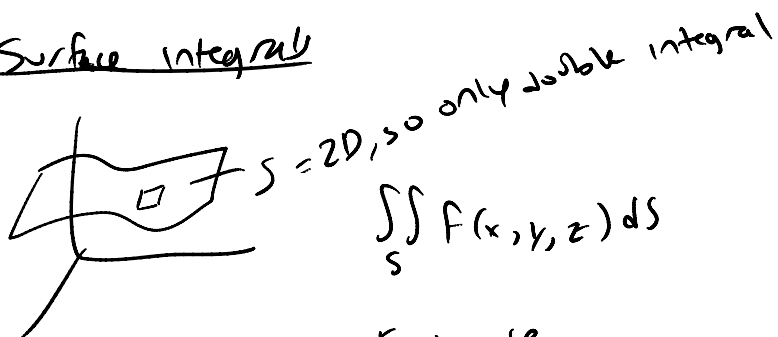
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

but we want dot product
w/ $\frac{\partial \vec{r}}{\partial w}$, so replace $\hat{i}, \hat{j}, \hat{k}$
with that.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right| = r^2 \sin \phi : \quad \left| \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)} \right| = \frac{1}{r^2 \sin \phi}$$

Surface integral



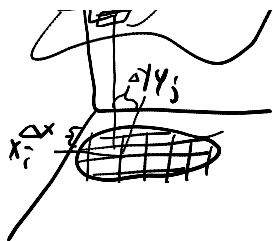
$$\iint_S F(x, y, z) dS$$

First case

Surface S is defined by the graph of a function

$$S: z = \phi(x, y) \quad (x, y) \in R$$

$$f(x, y, z) (dS) \rightarrow ?$$

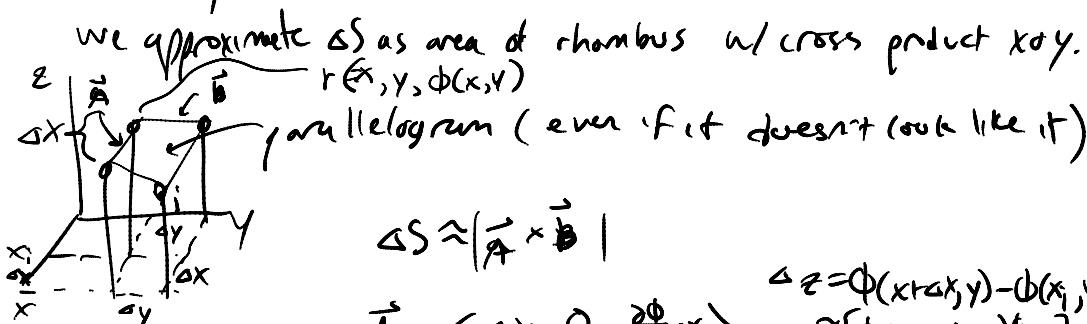


$$\iint_S f(x,y,z) dS = \sum_{x_i} \sum_{y_j} f(x_i, y_j, \phi(x_i, y_j)) \Delta S_{ij}$$

$(x_i, y_j) \in R$

$\Delta S \neq \Delta x \Delta y$ b/c if it's slanted, it's diff.

$$\Delta S \geq \Delta x \Delta y$$



$$\Delta S \approx |\vec{A} \times \vec{B}|$$

$$\vec{A} = \left\langle \Delta x, 0, \frac{\partial \phi}{\partial x} \Delta x \right\rangle$$

$$\begin{aligned} \Delta z &= \phi(x+\Delta x, y) - \phi(x, y) \\ &\approx \left[\phi(x, y) + \frac{\partial \phi}{\partial x} \Delta x \right] - \phi(x, y) \\ &= \frac{\partial \phi}{\partial x} \Delta x \end{aligned}$$

If the surface was flat, $\frac{\partial \phi}{\partial x}$ would be 0.

$$\vec{B} = \left\langle 0, \Delta y, \frac{\partial \phi}{\partial y} \Delta y \right\rangle$$

$$|\vec{A} \times \vec{B}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & \frac{\partial \phi}{\partial x} \Delta x \\ 0 & \Delta y & \frac{\partial \phi}{\partial y} \Delta y \end{vmatrix}$$

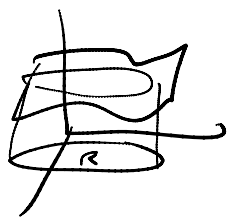
$$= -\hat{i} \phi_x \Delta x \Delta y - \hat{j} (\phi_y \Delta x \Delta y) + \hat{k} (\Delta x \Delta y)$$

$$\langle -\phi_x, -\phi_y, 1 \rangle \Delta x \Delta y$$

↑
normal to surface

$$\Delta S \hat{n} \quad ; \quad \Delta S = \sqrt{\phi_x^2 + \phi_y^2 + 1} \Delta x \Delta y$$

$$\iint_S f(x,y,z) dS = \iint_R f(x,y, \phi(x,y)) \sqrt{1 + \phi_x^2 + \phi_y^2} dx dy$$

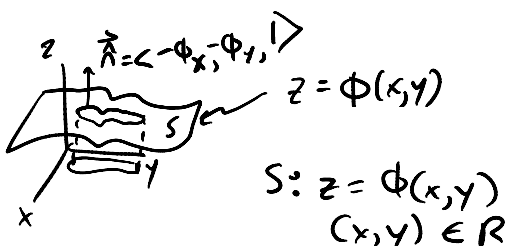


If you want to find surface area, you can do it this way.

Surface Integrals

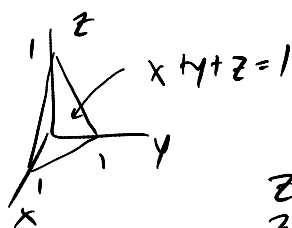
Monday, January 29, 2007
10:04 AM

Surface Integrals

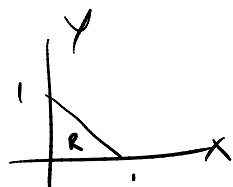
$$\iint_S f(x, y, z) dS$$


$\vec{n} = \langle -\phi_x, -\phi_y, 1 \rangle$
 $z = \phi(x, y)$
 $S: z = \phi(x, y)$
 $(x, y) \in R$

$$\iint_S f(x, y, \phi(x, y)) |\vec{n}| dA = \iint_R f(x, y, \phi(x, y)) \sqrt{1 + \phi_x^2 + \phi_y^2} dA$$



$z = \phi(x, y)$
 $z = 1 - x - y$
 $\phi_x = -1$
 $\phi_y = -1$
 $\vec{n} = \langle 1, 1, 1 \rangle$



Surface Area = $\iint_S dS$

$$= \iint_R \sqrt{1 + \phi_x^2 + \phi_y^2} dA$$

$$= \sqrt{3} \iint_R dA$$

Area of Δ

$$= \frac{\sqrt{3}}{2}$$

$$ax + by + cz = d$$

When y and z are zero, $= 1$

$$a \cdot 1 = d$$

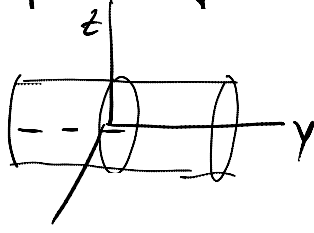
$$\langle 1, 0, 0 \rangle$$

$$\langle 0, 1, 0 \rangle$$

$$\langle 0, 0, 1 \rangle$$

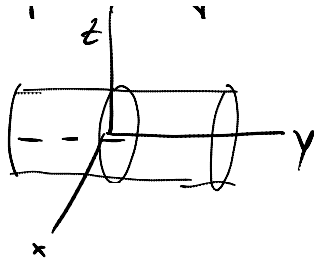
ex2

Compute surface area of cylinder $x^2 + z^2 = a^2$ $-a \leq y \leq a$
in xz plane, independent of y .



$$\text{Area} = \iint_S dS = \iint \sqrt{1 + \phi_x^2 + \phi_y^2} dA$$

Bring in factor of 2 b/c
the top surface is same as

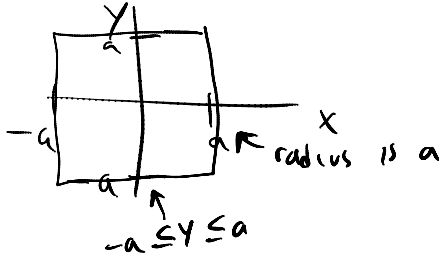


$$\text{Area} = \iint dS = \iint \sqrt{1 + \phi_x^2 + \phi_y^2} dA$$

Bring in factor of 2 b/c the top surface is same as bottom.

$$A = \iint_S dS = 2 \iint_R \sqrt{1 + \phi_x^2 + \phi_y^2} dA_x$$

Region is rectangle.



$$z = \phi(x, y) = \sqrt{a^2 - x^2}$$

$$\phi_y = 0$$

$$\phi_x = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)$$

$$SA = 2 \iint_R \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$\sqrt{\frac{a^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}$$

$$SA = 2a \iint_{-a}^a \sqrt{\frac{1}{a^2 - x^2}} dy dx = (2a)(2a) \int_{-a}^a \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$-a = a \sin \theta$$

$$-1 = \sin \theta$$

$$-\frac{\pi}{2} = \theta$$

$$a = a \sin \theta$$

$$1 = \sin \theta$$

$$\frac{\pi}{2} = \theta$$

$$x = a \sin \theta$$

$$dx = a \cos \theta d\theta$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta$$

$$SA = 4a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$SA = 4a^2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 4a^2 \pi$$

Check: $SA = \text{circumference} \times \text{height} = (2\pi a)(2a) = 4a^2 \pi$

Surface Area

$$\iint_S f(x, y, z) dS \leftarrow ?$$

$\nearrow \cos \theta$
 $\nwarrow \sin \theta$
 $\Phi(r, \theta)$
 $z = \Phi(r, \theta)$

$$dS = \sqrt{1 + \phi_x^2 + \phi_y^2} dx dy$$

$$z = \phi(x, y) = \phi(r \cos \theta, r \sin \theta) = \Phi(r, \theta) = z$$

$$\Phi(r, \theta)$$



$$\phi(x, y)$$

express ϕ_x & ϕ_y in terms of Φ_r & Φ_θ

$$z = \phi(x, y) = \Phi(r, \theta)$$

$$x = r \cos \theta$$

$$dx dy = r dr d\theta$$

$$r^2 = x^2 + y^2$$

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \leftarrow r \cos \theta = \cos \theta$$

$$2r \frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta$$

$$y = r \sin \theta$$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \theta} \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \theta} \frac{\partial \theta}{\partial y}$$

$$\tan \theta = \frac{y}{x}$$

$$\sec^2 \theta \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{r \sin \theta}{r^2 \cos^2 \theta} (\cos^2 \theta)$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}$$

$$\sec^2 \theta \frac{\partial \theta}{\partial y} = \frac{1}{x}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$$

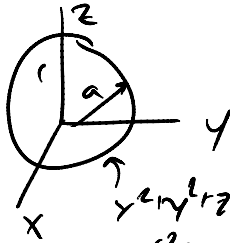
$$ds = \sqrt{1 + \left(\Phi_r \cos \theta - \Phi_\theta \frac{\sin \theta}{r} \right)^2 + \left(\Phi_r \sin \theta + \Phi_\theta \frac{\cos \theta}{r} \right)^2} r dr d\theta$$

$$ds = \sqrt{1 + \Phi_r^2 \cos^2 \theta - 2 \Phi_r \Phi_\theta \frac{\cos \theta \sin \theta}{r} + \Phi_\theta^2 \frac{\sin^2 \theta}{r^2} + \Phi_r^2 \sin^2 \theta + 2 \Phi_r \Phi_\theta \frac{\sin \theta \cos \theta}{r} + \Phi_\theta^2 \frac{\cos^2 \theta}{r^2}} r dr d\theta$$

$$ds = \sqrt{1 + \Phi_r^2 + \frac{1}{r^2} \Phi_\theta^2} r dr d\theta$$

which is a lot like $\sqrt{1 + \phi_x^2 + \phi_y^2}$

Use cylindrical coordinates to calculate surface area of a sphere of radius a



$$SA = \iint_S dS = \iint_a \sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} r dr d\theta$$

$$z = \Phi(r, \theta)$$

$$r^2 + z^2 = a^2$$

$$\sqrt{z^2} = \sqrt{a^2 - r^2}$$

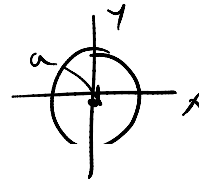
$$z = \pm \sqrt{a^2 - r^2}$$

use symmetry

(top hemisphere =

bottom) $2(\text{integral}) = SA$

$$\Phi = z = \sqrt{a^2 - r^2}$$



$$\Phi_r = \frac{1}{2}(a^2 - r^2)^{-\frac{1}{2}}(-2r) = -\frac{r}{\sqrt{a^2 - r^2}}$$

$$\sqrt{1 + \left(-\frac{r}{\sqrt{a^2 - r^2}}\right)^2 + \frac{1}{r^2}(0)} = \sqrt{\frac{a^2}{a^2 - r^2}} = \frac{a}{\sqrt{a^2 - r^2}}$$

$$SA = 2a \int_0^{2\pi} \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr d\theta$$

$$SA = 2a 2\pi \int_0^a \frac{r}{\sqrt{a^2 - r^2}} \quad \text{or just use } \int_0^a \Phi_r dr \text{ b/c it worked out that way.}$$

$$2a 2\pi \left[-\sqrt{a^2 - r^2} \right]_0^a$$

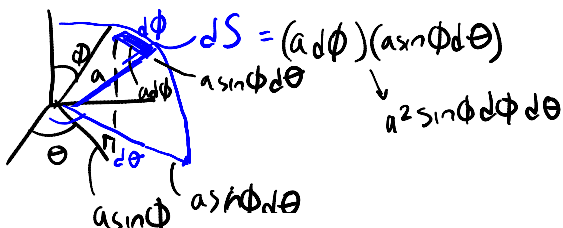
$$4\pi a [a] = 4\pi a^2$$

Surface Integral over a sphere or part of a sphere of radius a

$$dS = a^2 \sin\phi d\phi d\theta$$

compare w/ (c.f.) $dV = R^2 \sin\phi dr d\phi d\theta$

$\circledast a^2$ b/c radius doesn't change

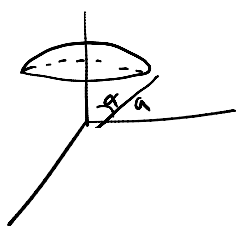


$$2\pi \pi$$

$$\begin{aligned}
 SA &= \iint_S ds = \int_0^\pi \int_0^{2\pi} a^2 \sin \phi d\phi d\theta \\
 &= 2\pi a^2 (-\cos \phi) \Big|_0^\pi \\
 &= 2\pi a^2 (2) \\
 &= 4\pi a^2
 \end{aligned}$$

This only works for a sphere

Compute surface area and center of mass of a spherical cap with $R=a$, $0 \leq \phi \leq \alpha$, density $= \delta_0$ (mass/area) such small thickness for cap that density is mass/area.



$$SA = \int_0^\alpha \int_0^{2\pi} a^2 \sin \phi d\phi d\theta$$

$$SA = 2a^2\pi (-\cos \phi) \Big|_0^\alpha$$

$$SA = 2\pi a^2 (-\cos \alpha - 1) \text{ flip bounds}$$

$$SA = 2\pi a^2 (1 - \cos \alpha)$$

$$c.m. = (\bar{x}, \bar{y}, \bar{z})$$

by symmetry, c.m. on z axis. $\bar{x}, \bar{y} = 0$

$$\bar{z} = \frac{1}{M} \iint_S z \delta_0 ds \quad \text{has units of mass}$$

$$M = SA \delta_0 = 2\pi a^2 (1 - \cos \alpha) \delta_0$$

$$z = a \cos \phi$$

$$\bar{z} = \frac{1}{M} \delta_0 \int_0^\alpha \int_0^{2\pi} a \cos \phi \sin \phi d\phi d\theta$$

$$\bar{z} = \frac{a \sin^2 \alpha}{2(1 - \cos \alpha)} = \frac{a}{2} \frac{(1 - \cos^2 \alpha)}{(1 - \cos \alpha)} = \frac{a}{2} (1 + \cos \alpha)$$

$$\alpha \rightarrow 0, \bar{z} \rightarrow a$$

$$\alpha \rightarrow \pi, \bar{z} \rightarrow 0$$

$$\alpha \rightarrow \frac{\pi}{2}, \bar{z} \rightarrow \frac{a}{2}$$

Surface integrals in general coordinates

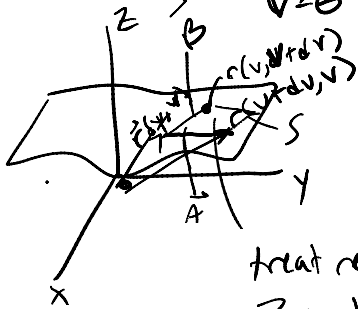
Just as a curve in 3D can be defined parametrically, by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, a surface in 3D can be defined parametrically by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

ex

$$\textcircled{1} \text{ Cartesian } \begin{matrix} u=x \\ v=y \end{matrix} \quad \vec{r} = \langle x, y, \phi(x,y) \rangle$$

② cylindrical, polar $u=r$ $\vec{r} = \langle r \cos \theta, r \sin \theta, \Phi(r, \theta) \rangle$
 $v=\theta$

③ spherical $u=\phi$ $\vec{r} = \langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi \rangle$
 $v=\theta$



$$\vec{r}(u+du, v) = \vec{r}(u, v) + \frac{d\vec{r}}{du} du$$

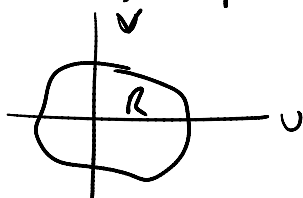
treat region like a $\square$ so dA is cross product of 2 vectors

$$ds = |\vec{A} \times \vec{B}|$$

$$\vec{A} = \frac{\partial \vec{r}}{\partial u} du$$

$$\vec{B} = \frac{\partial \vec{r}}{\partial v} dv$$

Surface given parametrically by $\vec{r} = \langle x(u,v), y(u,v), z(u,v) \rangle$



$$\iint_S f(x,y,z) dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$$

ex1

Surface $z = \Phi(x,y)$

$$u = x$$

$$v = y$$

$$z = \Phi(x,y)$$

$$\vec{r} = \langle x, y, \Phi(x,y) \rangle$$

$$dS = \frac{\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right|}{\sqrt{1 + \Phi_x^2 + \Phi_y^2}} dx dy$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \Phi_x \\ 0 & 1 & \Phi_y \end{vmatrix}$$

$$\frac{\partial \vec{r}}{\partial x} = \langle 1, 0, \Phi_x \rangle$$

$$\frac{\partial \vec{r}}{\partial y} = \langle 0, 1, \Phi_y \rangle$$

$$\hat{i}(-\Phi_x) - \hat{j}(\Phi_y) + \hat{k}(1)$$

$\langle -\Phi_x, -\Phi_y, 1 \rangle$ pointing normal to the surface.

$$z = \Phi(r, \theta)$$

$$u = r$$

$$v = \theta$$

$$\vec{r} = \langle r \cos \theta, r \sin \theta, \Phi(r, \theta) \rangle$$

$$dS = \left| \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} \right| dr d\theta = \sqrt{1 + \Phi_r^2 + \frac{1}{r^2} \Phi_\theta^2} r$$

$$\frac{\partial \vec{r}}{\partial r} = \langle \cos \theta, \sin \theta, \Phi_r \rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} = \langle -r \sin \theta, r \cos \theta, \Phi_\theta \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & \Phi_r \\ -r \sin \theta & r \cos \theta & \Phi_\theta \end{vmatrix} = \hat{i}(\Phi_\theta \sin \theta - \Phi_r r \cos \theta) - \hat{j}(\Phi_\theta \cos \theta + \Phi_r r \sin \theta) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta)$$

$$\begin{vmatrix} \cos\theta \sin\theta & \phi_r \\ -r\sin\theta & r\cos\theta \\ \phi_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta \sin\theta & \phi_r \\ -r\sin\theta & r\cos\theta \end{vmatrix} + \phi_\theta (r\cos^2\theta + r\sin^2\theta)$$

$$S = \sqrt{(\phi_\theta \sin\theta - \phi_r \cos\theta)^2 + (\phi_\theta \cos\theta + \phi_r \sin\theta)^2 + r^2}$$

$$dS = \int \sqrt{\phi_\theta^2 \sin^2\theta - 2r\phi_r\phi_\theta \sin\theta \cos\theta + r^2 \phi_r^2 \cos^2\theta + \phi_\theta^2 \cos^2\theta + 2r\phi_r\phi_\theta \sin\theta \cos\theta + r^2 \phi_r^2 \sin^2\theta + r^2} d\theta$$

$$= \sqrt{\phi_\theta^2 + r^2 \phi_r^2 + r^2} d\theta$$

$$= \sqrt{\frac{r^2}{r^2} \phi_\theta^2 + r^2 \phi_r^2 + r^2} d\theta$$

$$= \sqrt{\frac{1}{r^2} \phi_\theta^2 + \phi_r^2 + 1} d\theta$$

ex 3

Surface part of sphere of radius a.

$$u = \phi$$

$$v = \theta$$

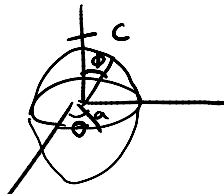
$$\vec{r} = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$$

$$dS = \left| \frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} \right| d\phi d\theta \quad \text{should} = a^2 \sin\phi d\phi d\theta$$

Ellipsoid HW prob (4)

$$\text{Surface area of ellipsoid } \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \quad c \leq a$$

$$SA = \iint_S dS$$



$$u = \phi$$

$$v = \theta$$

$$\vec{r} = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, c \cos\phi \rangle$$

See that equation is satisfied for values of ϕ & θ

$$0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

$$\iint_R \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = \sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^2\phi$$

$$\sin^2 \phi + \cos^2 \phi = 1$$

So that works. $\checkmark$

$$\frac{\partial \vec{r}}{\partial \phi} = \langle a \cos \phi \cos \theta, a \cos \phi \sin \theta, -c \sin \phi \rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} = \langle -a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -c \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$\hat{i}(-a c \sin^2 \phi \cos \theta) + \hat{j}(a c \sin^2 \phi \sin \theta) + \hat{k}(a^2 \sin \phi \cos \phi \cos^2 \theta + a^2 \sin \phi \cos \phi \sin^2 \theta)$$

take magnitude

$$\sqrt{(a^2 c^2 \sin^4 \phi \cos^2 \theta) + (a^2 c^2 \sin^4 \phi \sin^2 \theta) + (a^4 \sin^2 \phi \cos^2 \phi)}$$

$$\sqrt{a^2 c^2 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi} \, d\phi d\theta$$

$$= a c \sin \phi \sqrt{\sin^2 \phi + \frac{a^2}{c^2} \cos^2 \phi} \, d\phi d\theta$$

$$\text{check } a=c, \quad \sin^2 \phi + \cos^2 \phi = 1$$

$$a^2 \sin \phi \, d\phi d\theta \text{ is that's right. } \checkmark$$

Surface Area of ellipsoid

$$a c \int_0^{2\pi} \int_0^\pi \sin \phi \sqrt{\sin^2 \phi + \frac{a^2}{c^2} \cos^2 \phi} \, d\phi d\theta$$

$$2\pi a c \int_0^\pi \sin \phi \sqrt{\sin^2 \phi + \frac{a^2}{c^2} \cos^2 \phi} \, d\phi$$

$$(1 - \cos^2 \phi) + \frac{a^2}{c^2} \cos^2 \phi$$

$$1 - \cos^2 \phi \left(1 - \frac{a^2}{c^2}\right)$$

$$u = \cos \phi$$

$$du = -\sin \phi \, d\phi$$

$$-2\pi a c \int_1^{-1} \sqrt{1 - u^2 \left(1 - \frac{a^2}{c^2}\right)} \, du \quad c \leq a, \text{ so}$$

$$2\pi a c \int_0^\pi \sqrt{1 + u^2 \left(\frac{a^2}{c^2} - 1\right)} \, d\phi d\theta$$

just kidding

$$2\pi a c \int_0^\pi \sqrt{1 + \cos^2 \phi \left(\frac{a^2}{c^2} - 1\right)} \, d\phi d\theta \quad \sqrt{\frac{a^2}{c^2} - 1} \cos \phi = u$$

$$-\sqrt{\frac{a^2}{c^2} - 1} \sin \phi \, d\phi = du$$

$$SA = \frac{2\pi ac}{\sqrt{\frac{a^2}{c^2}-1}} \int_{-\sqrt{\frac{a^2}{c^2}-1}}^{\sqrt{\frac{a^2}{c^2}-1}} \sqrt{1+u^2} du$$

$$u = \tan \alpha$$

$$du = \sec^2 \alpha d\alpha$$

$$\int \sec^3 \alpha d\alpha$$

$$\sec^2 \alpha d\alpha = du$$

$$\sec \alpha = \sqrt{1+u^2}$$

$$= \sec \alpha \tan \alpha - \int \sec \alpha \tan^2 \alpha d\alpha$$

$$-\sqrt{\frac{a^2}{c^2}-1} \sinh \phi d\phi = du$$

$$1+u^2 = 1+\tan^2 \alpha$$

$$1+u^2 = \sec^2 \alpha$$

$$u = \tan \alpha$$

$$du = \sec^2 \alpha d\alpha$$

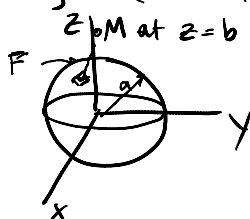
Problem

- Show that the gravitational attraction on a point mass m by a spherical shell of uniform density & mass m is
- (1) 0 if point mass m is inside the spherical shell
 - (2) equivalent to assuming the same mass m is concentrated at center of sphere if M is outside sphere



Set up coordinate system s.t. origin on center of sphere & point mass M on z axis.

w.l.o.g. (without loss of generality).



Result depends on whether $b > a$, $b < a$.
 $\uparrow$ outside sphere $\uparrow$ inside sphere

$$\vec{F} = \frac{GMm}{r^2} \hat{r}$$

let mass density (in units mass/area)
 $= \sigma_a$ related to m

$$m = \sigma_a (4\pi a^2)$$

$\uparrow$ area of sphere

$$\Delta m = \sigma_a \Delta S$$

Δm at location $(x, y, z) = (a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi)$

M at $(x, y, z) = (0, 0, b)$

$$\vec{r} = \langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi - b \rangle$$

$\vec{r}$ is a vector from b to that little patch where F goes.

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi - b \rangle}{\sqrt{a^2 \sin^2 \phi + a^2 \cos^2 \phi - 2ab \cos \phi + b^2}}$$

$$\frac{\langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi - b \rangle}{(a^2 - 2ab \cos \phi + b^2)^{3/2}}$$

$$\vec{F} = GM \sigma_a^2 \int_0^{2\pi} \int_0^\pi \frac{\langle a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi - b \rangle}{(a^2 - 2ab \cos \phi + b^2)^{3/2}} \sin \phi d\phi d\theta$$

$$\Delta S \rightarrow dS = a^2 \sin \phi d\phi d\theta \quad \Delta m = \sigma_a \Delta S$$

$$\vec{F} = \langle F_x, F_y, F_z \rangle$$

The attraction will be in the z direction

$$F_x = GM \sigma_a a^3 \int_0^{2\pi} \int_0^\pi \frac{\sin \phi \cos \theta}{(a^2 + b^2 - 2ab \cos \phi)^{3/2}} d\phi d\theta$$

only b/c of the placement of the pt.

$$\left(\frac{\sin \phi}{r}\right)^2 \sin \phi \Big|_0^\pi = 0$$

$$F_y = G M \sigma_0 a^3 \int_0^{2\pi} \int_0^\pi \frac{\sin^2 \theta \sin \phi}{(a^2 + b^2 - 2ab \cos \theta)^{3/2}} d\phi d\theta$$

$$= 0$$

$$F_z = 2\pi G M \sigma_0 a^2 \int_0^\pi \frac{(a \cos \theta - b) \sin \theta}{(a^2 + b^2 - 2ab \cos \theta)^{3/2}} d\theta d\phi$$

$$u = a^2 + b^2 - 2ab \cos \phi$$

$$du = 2ab \sin \phi d\phi$$

$$a \cos \phi = \frac{a^2 + b^2 - u}{2b}$$

$$\frac{a^2 + b^2 - u - 2b^2}{2b}$$

$$\frac{1}{\sin \phi} \int \frac{(a^2 + b^2 - u)}{u^2} du$$

GR

$$u = \cos \phi$$

$$du = -\sin \phi d\phi$$

$$\int_{u=1}^{u=-1} \frac{(au-b)}{(a^2 + b^2 - 2abu)^{3/2}} du = \frac{\partial}{\partial b} \int_{-1}^1 \frac{1}{(a^2 + b^2 - 2ab u)^{1/2}} du$$

$$= \frac{\partial}{\partial b} \int_{-1}^1 \left[\frac{1}{(a^2 + b^2 - 2ab u)^{1/2}} \right] du = \frac{\partial}{\partial b} \left[\frac{2}{-2ab} \left(\sqrt{a^2 + b^2 - 2ab u} \right) \right]_{-1}^1$$

$$= \frac{\partial}{\partial b} \left[\left(-\frac{1}{ab} \sqrt{a^2 + b^2 - 2ab} \right) - \left(-\frac{1}{ab} \sqrt{a^2 + b^2 + 2ab} \right) \right]$$

$$= \frac{\partial}{\partial b} \left[-\frac{1}{ab} (a-b) + \frac{1}{ab} (a+b) \right]$$

$$= \frac{\partial}{\partial b} \left[\frac{a+b}{ab} - \frac{a-b}{ab} \right]$$

$(a-b) = \text{pos}$ if particle is

inside sphere

$|a-b| = \text{neg}$ if particle is outside

$$= 2\pi G M \sigma_0 a^2 \begin{cases} \frac{\partial}{\partial b} \left[\frac{a+b-(a-b)}{ab} \right] & a > b \\ \frac{\partial}{\partial b} \left[\frac{a+b-(b-a)}{ab} \right] & a < b \end{cases}$$

$$= 2\pi G M \sigma_0 a^2 \begin{cases} \frac{\partial}{\partial b} \left[\frac{2}{a} \right] \\ \frac{\partial}{\partial b} \left[\frac{2}{b} \right] \end{cases}$$

$$\frac{\partial}{\partial b} \left[\frac{2}{a} \right] = 0$$

$$\frac{\partial}{\partial b} \left[\frac{2}{b} \right] = -\frac{2}{b^2}$$

$$-4\pi G M \sigma_0 a^2 \left(\frac{1}{b^2} \right) \quad a < b$$

mass of spherical shell, m

$$- \frac{GMm}{b^2}$$

$b^2 \rightarrow$ distance from center of spherical shell to point mass.

$$\int_{-1}^1 \frac{\partial I}{\partial b} du = \frac{\partial}{\partial b} \int_{-1}^1 I du$$

Vector Field

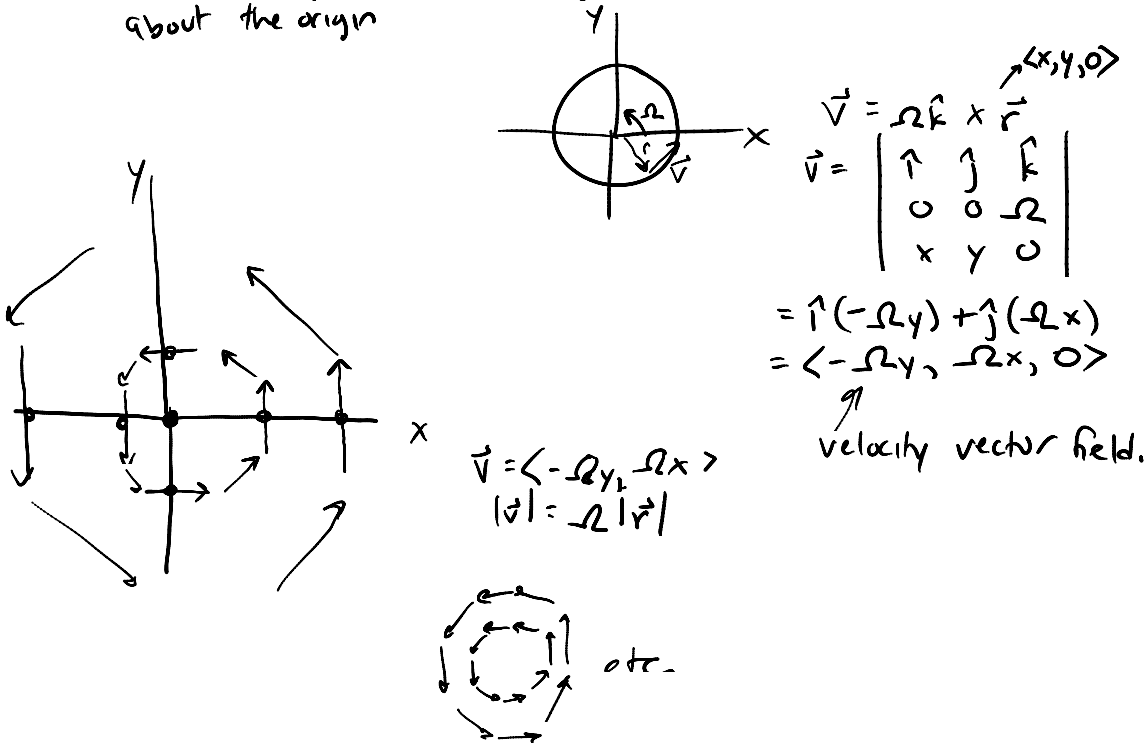
Associates a vector with each point in space

e.g. $\vec{V}(x, y, z) = x^2y \hat{i} + xy^2 \hat{j} + z^2 \hat{k}$

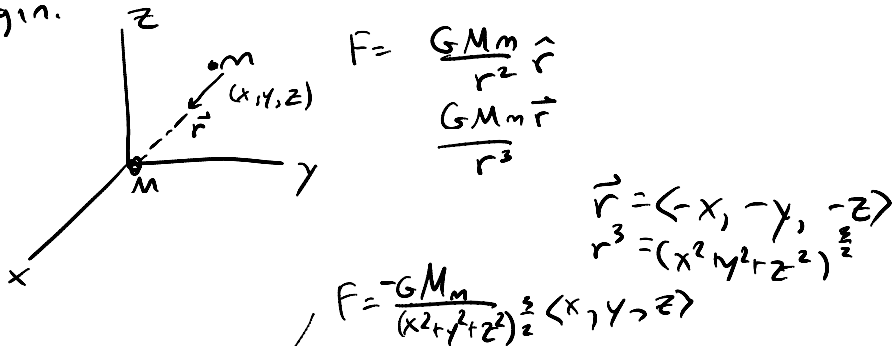
Sometimes referred to as vector value functions

ex 2D

Velocity of pts on a rotating disk w/ angular speed Ω rotating about the origin



ex Gravitational force field on a test mass m if mass M is at origin.



Vectors diverge as you go to origin.

Further away, shrinks in length.

$= \nabla \phi = \langle \phi_x, \phi_y, \phi_z \rangle$ Gradient of a potential

$$P = -\frac{GMm}{r} \stackrel{?}{=} \frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

$$\Phi_x = \frac{1}{2} (x^2+y^2+z^2)^{-\frac{3}{2}} GMm (2x)$$

$$\frac{GMm x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \quad \text{And this is } x \text{ component of the vector field.}$$

$$\vec{F} = -\frac{GMm}{(x^2+y^2+z^2)^{\frac{3}{2}}} \langle x, y, z \rangle$$

$$\nabla \Phi, \quad \Phi = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

- ① Potential, Φ , is not unique. We can always add a constant to it. & still have $\nabla \Phi = \vec{F}$
- ② Φ is called a scalar potential for $\vec{F}$ if $\vec{F} = \nabla \Phi$
- ③ Φ is equal to (potential energy of the mass M)
neg b/c as you go further away, PE increases
- ④ $V = -\frac{\Phi}{m}$ is called gravitational potential and is potential energy per unit mass

$$\begin{aligned} \vec{V} &= \langle -\Omega y, -\Omega x \rangle \\ &\stackrel{?}{=} \nabla \Phi \\ &= \langle \Phi_x, \Phi_y \rangle \end{aligned}$$

$$\begin{aligned} \Phi &= -\Omega xy \quad \times \\ \Phi_x &= -\Omega y \quad \Phi_y \neq -\Omega x \end{aligned}$$

So you cannot always find a potential

Note: Not all vector fields have a scalar potential

EX 2D test

$$\begin{aligned} \vec{V} = \langle V_1, V_2 \rangle &\stackrel{?}{=} \nabla \Phi = \langle \Phi_x, \Phi_y \rangle \\ V_1 &= \Phi_x \\ V_2 &= \Phi_y \end{aligned}$$

$$\text{or check whether } \frac{\partial V_1}{\partial y} \stackrel{?}{=} \frac{\partial V_2}{\partial x}$$

$$\vec{V} = \langle -\Omega y, -\Omega x \rangle$$

$$\frac{\partial V_1}{\partial y} = -\Omega$$

$$\frac{\partial V_2}{\partial x} = \Omega$$

$\neq$ potential

$$\begin{aligned} \Phi_{xy} & \quad \Phi_{yx} \\ & \text{and these should =} \end{aligned}$$

Practice Midterm

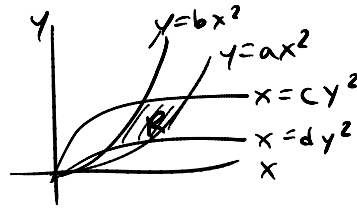
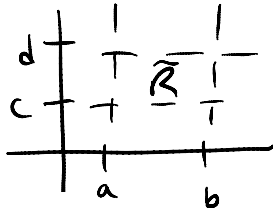
Tuesday, February 06, 2007
9:59 AM

Practice Midterm

2. $b > a > 0$ $d > c > 0$

$$u = \frac{y}{x^2}$$

$$v = \frac{x}{y^2}$$



$$\text{Area} = \iint_R dx dy$$

$$\iint_R \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

$$\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{2y}{x^2} & \frac{1}{y^2} \\ \frac{1}{x} & -\frac{2x}{y^2} \end{vmatrix} = \frac{3}{x^2 y^2}$$

$$\iint_R \frac{x^2 y^2}{3} du dv$$

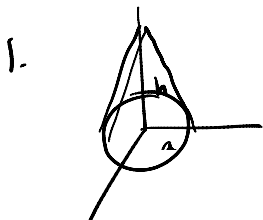
$$A = \frac{1}{3} \int_c^d \int_a^b \frac{1}{v^2} du dv$$

$$= \frac{1}{3} \left[\int_c^d \frac{1}{v^2} dv \cdot \int_a^b \frac{1}{v^2} du \right] \quad \text{b/c bounds are constants}$$

$$u \cdot v = \frac{1}{xy}$$

$$= \frac{1}{3} \left[\left[-\frac{1}{v} \right]_c^d \cdot \left[-\frac{1}{v} \right]_a^b \right]$$

$$= \frac{1}{3} \left(\frac{1}{c} - \frac{1}{d} \right) \cdot \left(\frac{1}{a} - \frac{1}{b} \right) = \text{this pos. quantity}$$



density = ρ
center of mass, moment of inertia about z axis.

$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$\bar{z} = \frac{1}{M} \iiint \rho z dV$$

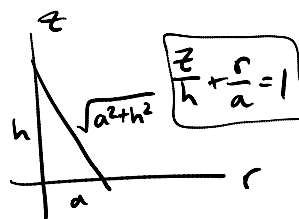
$$\bar{z} = \frac{1}{M} \iiint_V \rho z \, dV$$

$$V = \frac{1}{3} \pi a^2 h$$

you get the
mass this way

$$\rho = \frac{M}{(\frac{1}{3} \pi a^2 h)}$$

$$\frac{\rho}{3} \pi a^2 h = M$$



$$z = (1 - \frac{r}{a})h$$

$$z = h - \frac{h}{a}r$$

$$\bar{z} = \frac{3}{\pi a^2 h} \iiint_V z \, dV$$

$\underbrace{r \, dr \, d\theta \, dz}_{r \, dr \, d\theta \, dz}$

$$\bar{z} = \frac{3}{\pi a^2 h} \int_0^{2\pi} \int_0^a \int_0^{h-\frac{h}{a}r} r z \, dz \, dr \, d\theta$$

$$\bar{z} = \frac{3}{\pi a^2 h} \int_0^{2\pi} \int_0^a r \frac{(h - \frac{h}{a}r)^2}{2} \, dr \, d\theta$$

$$\bar{z} = \frac{3}{a^2 h} \int_0^a r (h - \frac{h}{a}r)^2 \, dr$$

$$\bar{z} = \frac{3}{a^2 h} \int_0^a h^2 r - \frac{2h^2}{a} r^2 + \frac{h^2}{a^2} r^3$$

$$\frac{3h}{a^2} \left(\frac{r^2}{2} - \frac{2r^3}{3a} + \frac{r^4}{4a^2} \right) \Big|_0^a$$

$$3h \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right)$$

$$\frac{3h}{12} = \left(\frac{h}{4} \right)$$

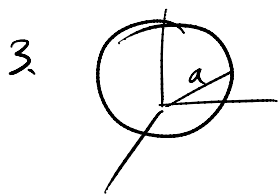
$$I = \iiint_V \rho \, d^2 \, dV$$

$\uparrow$
 r^2

$$I = \rho \int_0^{2\pi} \int_0^a \int_0^{h-\frac{h}{a}r} r^3 \, dz \, dr \, d\theta$$

$$= \frac{\pi}{10} \rho h a^4$$

I has mass-distance² Units.



$$\rho = k \sqrt{x^2 + y^2} = k R \sin \phi$$

$$M = \iiint \rho \, dV$$

$$R^2 \sin \phi \, dR \, d\phi \, d\theta$$

$$M = k \int_0^{2\pi} \int_0^{\pi} \int_0^1 R^3 \sin^2 \phi \, dR \, d\phi \, d\theta$$

$$M = k \frac{\pi^2}{1}$$

4. $z = xy$ in cyl. $x^2 + y^2 = 1$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx \, dy$$

$$dS = \sqrt{1 + y^2 + x^2} \, dx \, dy$$

$$dS = \sqrt{1 + r^2} \, r \, dr \, d\theta$$

$$SA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} \, r \, dr \, d\theta$$

$$\frac{2}{3} \pi (2\sqrt{2} - 1)$$

Line Integrals, Path Independent

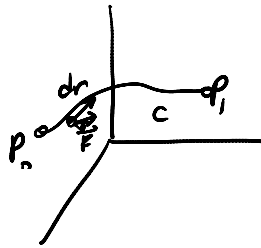
Thursday, February 08, 2007

9:52 AM

Line Integrals

$$W = \int_C \vec{F} \cdot d\vec{r}$$

Work = Force $\cdot$ $\overset{\substack{\uparrow \\ \text{direction you're moving}}}{r^1} \cdot d$ $\overset{\substack{\uparrow \\ \text{distance}}}{d}$
= Work done by force $\vec{F}$ and moving in particle along C from P_0 to P_1



$$\vec{dr} = \langle dx, dy, dz \rangle$$
$$\vec{F} = \langle F_1, F_2, F_3 \rangle = F(x, y, z)$$

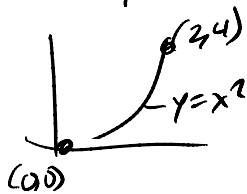
\ \ \ /
can be functions of position.

$$P_0 = (x_0, y_0, z_0)$$
$$P_1 = (x_1, y_1, z_1)$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_{x_0}^{x_1} F_1 dx + \int_{y_0}^{y_1} F_2 dy + \int_{z_0}^{z_1} F_3 dz$$

BUT don't Forget, $F_i = F(x, y, z)$, (x, y, z) points on C .

2-D example



$$C = y = x^2 \text{ from } P_0 = (0, 0) \text{ to } P_1 = (2, 4)$$

$$\vec{F} = \langle y, 2x \rangle$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \langle y, 2x \rangle \cdot \langle dx, dy \rangle$$

$$= \int_0^2 y dx + \int_0^4 2x dy$$

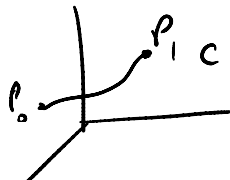
$$= \int_0^2 x^2 dx + \int_0^4 2\sqrt{y} dy$$

$$= \left[\frac{x^3}{3} \right]_0^2 + \left[\frac{4}{2} y^{\frac{3}{2}} \right]_0^4$$

$$= \frac{8}{3} + \frac{32}{5} = \left(\frac{40}{3} \right)$$

$$y = x^2$$
$$\sqrt{y} = x$$

A standard method for evaluating line integrals is to define C parametrically.



$$\vec{r} = \langle x(t), y(t), z(t) \rangle$$

$$t_0 \leq t \leq t_1$$

$\uparrow$ at P_0 $\uparrow$ at P_1

$$P_0 = \vec{r}(t_0)$$

$$P_1 = \vec{r}(t_1)$$

$$W = \int_C \vec{F} \cdot d\vec{r} \quad d\vec{r} \text{ is } \langle dx, dy, dz \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

change of variables

$$= \frac{d\vec{r}}{dt} dt$$

$$= \int_{t=t_0}^{t=t_1} \vec{F}(x(t), y(t), z(t)) \cdot \frac{d\vec{r}}{dt} dt$$

Integral over t variable.

ex

$$C = y = x^2$$

$$\vec{F} = \langle y, 2x \rangle$$

$$x = t$$

$$y = t^2$$

$$0 \leq t \leq 2$$

$$P_0 = (0, 0) \quad P_1 = (2, 4)$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^2 \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

$$\vec{F} = \langle y, 2x \rangle$$

$$\vec{F} = \langle t^2, 2t \rangle$$

$$\vec{r} = \langle t, t^2 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle 1, 2t \rangle$$

$$W = \int_0^2 \langle t^2, 2t \rangle \cdot \langle 1, 2t \rangle dt$$

$$W = \int_0^2 t^2 dt + \int_0^2 4t^2 dt$$

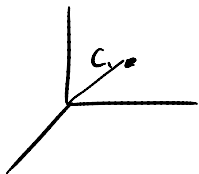
$$W = \left[\frac{t^3}{3} \right]_0^2 + \left[\frac{4t^3}{3} \right]_0^2 = \left(\frac{40}{3} \right)$$

STAR TREK

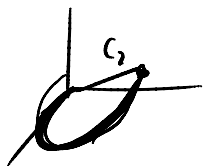
ex 2 vector field
 $\vec{F} = \langle x - z, 1 - xy, y \rangle$

compute $\int_{C_1} \vec{F} \cdot d\vec{r}$ + $\int_{C_2} \vec{F} \cdot d\vec{r}$ where C_1 & C_2 go from the origin to $(1, 1, 1)$

C_1 straight line from $(0, 0, 0)$ to $(1, 1, 1)$



$C_2 \quad \vec{r} = \langle t, t^2, t^3 \rangle \quad 0 \leq t \leq 1$



Takes you to same ending pt. but isn't straight.

Straight line: $\langle t, t, t \rangle = \vec{r} \quad 0 \leq t \leq 1$

$\frac{d\vec{r}}{dt} = \langle 1, 1, 1 \rangle$

Curved line: $\langle 1, 2t, 3t^2 \rangle = \frac{d\vec{r}}{dt}$

$\vec{F} = \langle t - t^3, 1 - t^3, t^2 \rangle$ Curved line

$\vec{F} = \langle 0, 1 - t^2, t \rangle$ Straight line.

$\int_0^1 1 - t^2 + t \, dt$
 $\left[t - \frac{t^3}{3} + \frac{t^2}{2} \right]_0^1$
 $1 - \frac{1}{3} + \frac{1}{2} = \frac{7}{6}$

$\int_0^1 t - t^2 + 2t - 2t^4 + 3t^4 \, dt$
 $= \int_0^1 3t - t^2 + t^4 \, dt$
 $= \left[\frac{3t^2}{2} - \frac{t^3}{3} + \frac{t^5}{5} \right]_0^1$
 $= \frac{3}{2} - \frac{1}{3} + \frac{1}{5} = \frac{29}{30}$

The answer depends on the path we take.

Circumstances w/ answer independent of path

$\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$ depends on path taken from P_0 to P_1 .

For some F , the value of the line integral is independent of the path. Path independent

when $\vec{F}$ can be written as $\vec{F} = \nabla \phi$
↑
potential

These vector fields are called conservative vector fields.

Force fields give things like conservation of energy.

In this case, $\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = \phi(P_1) - \phi(P_0)$

$\vec{F} = \nabla \phi$ where $\phi = xz + y^2$ $0 \leq t \leq 1$
 $\vec{r} = \underset{C_1}{\langle t, t, t \rangle}$ $\vec{r} = \underset{C_2}{\langle t, t^2, t^3 \rangle}$

$\nabla \phi = \langle z, 2y, x \rangle$
 $\vec{F} = \langle t, 2t, t \rangle$

$\vec{F} = \langle t^3, 2t^2, t \rangle$

$\frac{d\vec{r}}{dt} = \langle 1, 1, 1 \rangle$

$\frac{d\vec{r}}{dt} = \langle 1, 2t, 3t^2 \rangle$

$\int_0^1 (t + 2t + t) dt$

$\int_0^1 (t^3 + 4t^3 + 3t^3) dt$

$\int_0^1 4t dt$

$\int_0^1 8t^3 dt$

$2t^2 \Big|_0^1 = 2$

$2t^4 \Big|_0^1 = 2$

Path independent

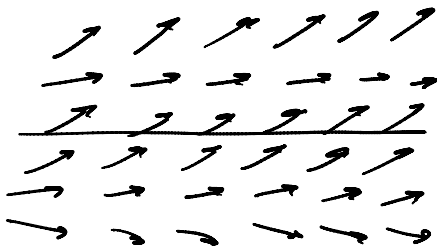
You can go ahead & check w/ $\phi(P_1) - \phi(P_0)$

b/c fundamental theorem

$$\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = \int_{P_0}^{P_1} \nabla \phi = \phi(P_1) - \phi(P_0)$$

Streamlines

Friday, February 09, 2007
10:00 AM



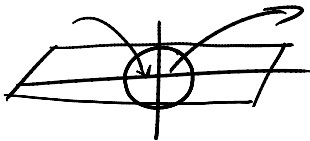
Streamlines

Suppose you have a vector
field $\vec{v} = \langle v_1, v_2, v_3 \rangle$

If you dropped a leaf in there you could predict where it goes.

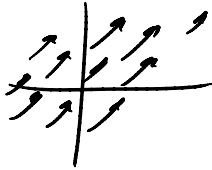
Let the streamline curve be defined by $\vec{r} = \langle x(t), y(t), z(t) \rangle$

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle v_1, v_2, v_3 \rangle = \vec{v}$$



Given the velocity, going to the streamline is difficult & sometimes impossible.

ex Take the velocity field $\vec{v} = \langle a, b, c \rangle$ constant vector.



$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle v_1, v_2, v_3 \rangle = \langle a, b, c \rangle$$

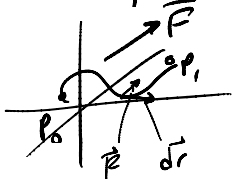
$$a = \frac{dx}{dt} \quad b = \frac{dy}{dt} \quad c = \frac{dz}{dt}$$

$$at + C = x \quad bt + D = y \quad ct + E = z$$

$$C = x_0 \quad D = y_0 \quad E = z_0$$

$$at + x_0 = x \quad bt + y_0 = y \quad ct + z_0 = z$$

Path Independence



$$\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = \int_{r=r(t_0)}^{r=r(t_1)} \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

Under certain circumstances value of
regardless of path chosen.

Path independence.

$\vec{F} = \nabla \phi$ ← potential function
conservative vector

$$\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} \text{ is constant}$$

Equivalent

Path independence of $\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r}$
is equivalent to $\vec{F}$ is conservative ($\vec{F} = \nabla \phi$)

$$\int_{p_0}^{p_1} \vec{F} \cdot d\vec{r} = \Phi(p) - \Phi(p_0)$$

$$\int_{p_0}^{p_1} \vec{F} \cdot d\vec{r} = \int_{p_0}^{p_1} \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_{t_0}^{t_1} \vec{\nabla} \phi \cdot \frac{d\vec{r}}{dt} dt$$

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\vec{r} = \langle x(t), y(t), z(t) \rangle \quad \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$\int_{t_0}^{t_1} \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

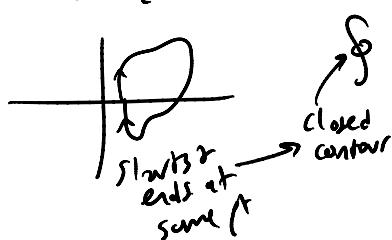
$$\int_{t_0}^{t_1} \left(\frac{\partial \phi}{\partial x} \frac{dx}{dt} + \frac{\partial \phi}{\partial y} \frac{dy}{dt} + \frac{\partial \phi}{\partial z} \frac{dz}{dt} \right) dt$$

$$\int_{t_0}^{t_1} \frac{d\phi}{dt} dt$$

$$\int_{t_0}^{t_1} d\phi = \phi(x(t_1), y(t_1), z(t_1)) - \phi(x(t_0), y(t_0), z(t_0))$$

Corollary

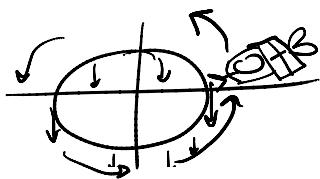
If $\int_{p_0}^{p_1} \vec{F} \cdot d\vec{r}$ is path independent then integral over any closed contour of $\vec{F} \cdot d\vec{r}$ is zero.



$$\oint \vec{F} \cdot d\vec{r} = 0 \quad \text{if} \quad \vec{F} = \vec{\nabla} \phi$$

$$\text{b/c} \quad \phi(p_0) - \phi(p_0) = 0$$

Gravity, electric fields, magnetism ... conservative!



no work done b/c you come back to same point.

These are all iff statements

Curl

| $\wedge$

$\oint_{\text{curl}}$

$\wedge$ |

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

rate of rotation

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \hat{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\vec{\nabla} \times \vec{F} = 0 \quad \text{irrotational vector field}$$

if a field is irrotational it's also conservative.

ex

$$\vec{F} = (x^2 + yz)\hat{i} + e^x\hat{j} + (z^3 \sin y)\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & e^x & z^3 \sin y \end{vmatrix}$$

$$\text{Curl} = \hat{i}(z^3 \cos y - 0) + \hat{j}(y - 0) + \hat{k}(e^x - z)$$

$$\int_{\gamma} \vec{F} \cdot d\vec{r}, \quad \vec{F} = \vec{\nabla} \phi, \quad \boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$$

$\vec{F}$ is irrotational (curl of $\vec{F} = 0$, $\vec{\nabla} \times \vec{F} = \vec{0}$) iff it is conservative

Proof of $\vec{F} = \vec{\nabla} \phi$ implies $\vec{\nabla} \times \vec{F} = \vec{0}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \hat{i} \left(\frac{\partial^2 \phi}{\partial z \partial y} - \frac{\partial^2 \phi}{\partial z \partial y} \right) + \hat{j} \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right) + \hat{k} \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \right)$$

$$= 0 + 0 + 0 = 0$$

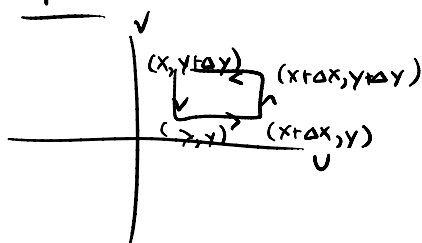
There's an interpretation of the curl as a local line integral
 curl $\rightarrow$ Local Line Integral.



$$\lim_{A \rightarrow 0} \frac{\oint_C \vec{F} \cdot d\vec{r}}{A} = (\vec{\nabla} \times \vec{F}) \cdot \hat{N}$$

If you take the curve C + shrink it to zero you get a line integral.

"Proof"



$$\frac{1}{A} \oint_C \vec{F} \cdot d\vec{r}$$

$$= \frac{1}{\Delta x \Delta y} \left\{ \int_{x+\Delta x}^{x+\Delta x} F_1(u, y) du + \int_y^{y+\Delta y} F_2(x+\Delta x, v) dv + \int_{x+\Delta x}^x F_1(u, y+\Delta y) du + \int_{y+\Delta y}^y F_2(x, v) dv \right\}$$

$$\begin{aligned} \frac{1}{A} \oint_C \vec{F} \cdot d\vec{r} &= \frac{1}{\Delta x \Delta y} \left\{ F_1(x, y) \Delta x + F_2(x+\Delta x, y) \Delta y \right. \\ &\quad \left. - F_1(x, y+\Delta y) \Delta x - F_2(x, y) \Delta y \right\} \\ &= \vec{\nabla} \times \vec{F} \end{aligned}$$

infinitely small Δx

Conservative Vector Fields

Monday, February 12, 2007
10:03 AM

Conservative Vector Fields - You can write the vector field as $\vec{\nabla}\phi$

$$(\exists \text{ } \wedge \text{ } \phi \text{ s.t. } \vec{F} = \nabla\phi)$$

there exists a scalar

$$\vec{F}(x,y,z)$$

The following 4 statements are equivalent:

i) $\int_C \vec{F} \cdot d\vec{r}$ is path independent

ii) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C .

iii) $\vec{F} = \nabla\phi$

iv) $\nabla \times \vec{F} = \vec{0}$ curl of F is 0

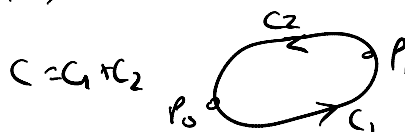
Given any you know the others are true

iii) $\Rightarrow$ iv) implies

we haven't shown that iv) $\Rightarrow$ iii)

i) $\Rightarrow$ ii)

ii) $\Rightarrow$ i)



$$0 = \oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{-C_2} \vec{F} \cdot d\vec{r}$$

And then it doesn't matter if $C_2 = C_1$ if it's (+) or (-), it still gives you the same thing. so ii) $\Rightarrow$ i)

i) $\Rightarrow$ ii) ?

(if we know iii), $\oint \nabla\phi = 0$ so we know iii) $\Rightarrow$ i)

Fundamental theorem of line integrals.

iv) $\Rightarrow$ iii) ?

i) $\Rightarrow$ iii)

show i) $\int \vec{F} \cdot d\vec{r}$ path independent $\Rightarrow$ ii) $\vec{F} = \nabla\phi$

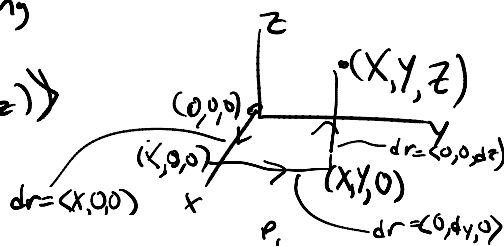
1) = 1(111)
 Show i) $\int \vec{F} \cdot d\vec{r}$ path independent $\Rightarrow$ ii) $\nabla \cdot \vec{F} = \nabla \phi$

Find ϕ explicitly by integrating

$$\vec{F} = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$$

$$p_0 = (0, 0, 0)$$

$$p_1 = (X, Y, Z)$$



$$\begin{aligned} \text{Let } \phi(X, Y, Z) &= \int_{p_0}^{p_1} \vec{F} \cdot d\vec{r} \\ &= \int_0^X F_1(x, 0, 0) dx + \int_0^Y F_2(X, y, 0) dy + \int_0^Z F_3(X, Y, z) dz \end{aligned}$$

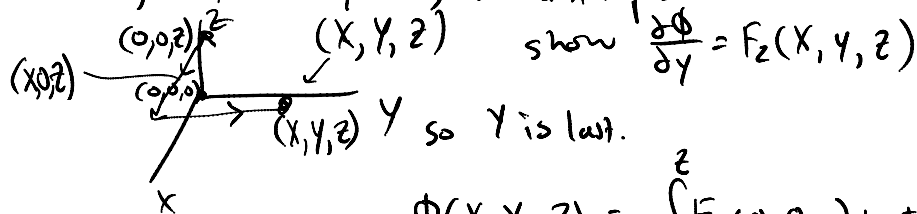
↑ depends on X
↑ depends on X=Y
↑ depends on X, Y, Z

Show that $\frac{\partial \phi}{\partial z} = F_3(X, Y, Z)$

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= \frac{\partial}{\partial z} \int_0^z F_3(X, Y, z) dz \\ &= F_3(X, Y, Z) \end{aligned}$$

We can't get the others w.r.t. z so it needs to be another way.

To get other components, use diff. path.



show $\frac{\partial \phi}{\partial y} = F_2(X, Y, Z)$

$$\begin{aligned} \phi(X, Y, Z) &= \int_0^Z F_3(0, 0, z) dz + \int_0^X F_1(x, 0, Z) dx \\ &\quad + \int_0^Y F_2(X, y, Z) dy \end{aligned}$$

Same thing when $\frac{\partial \phi}{\partial x}$.

$$0 + 0 + (X, Y, Z)$$

Then do same thing for $\frac{\partial \phi}{\partial x}$.

Work Energy Theorem

Thursday, February 15, 2007
9:57 AM

If $\nabla \times \vec{F}$ is 0, you can find a potential. If $F = \nabla \Phi$
you can show $\nabla \times \vec{F} = 0$
See handout

Work-energy theorem

$$F = \nabla \Phi$$

$$\Phi = -U$$

Suppose $\vec{F}$ is a conservative force + $F = -\nabla U$ then for a particle moving under the influence of F ($m\vec{a} = \vec{F}$), the energy
 $E = \frac{1}{2}mv^2 + U(\vec{r}) = K.E. + P.E.$

∇U points toward max. of U . $-\nabla U$ means pointing toward minimum of U .

is conserved.

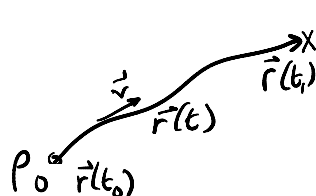
The change in kinetic energy is equal to the work done by the force.

To prove: let $\vec{r}(t)$ = position of particle at time t .
 $\vec{v}(t) = \frac{d\vec{r}}{dt}$ = velocity
 $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2}$ = acceleration = $\frac{\vec{F}}{m} = -\frac{\nabla U}{m}$

$$\text{Speed}^2 = v^2 = \vec{v} \cdot \vec{v} = |\vec{v}|^2$$

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2 \frac{d\vec{v}}{dt} \cdot \vec{v}$$

$$= 2 \vec{a} \cdot \vec{v}$$

 energy is the same at every pt. along the trajectory.

$$W = \int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = \int_{P_0}^{P_1} -\nabla U \cdot d\vec{r}$$

$$= -[U]_{P_0}^{P_1} = U(P_0) - U(P_1)$$

$$W = \int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = m \int_{t_0}^{t_1} (\vec{a} \cdot \vec{v}) dt = m \int_{t_0}^{t_1} \frac{1}{2} \frac{d}{dt}(v^2) dt$$

$$m \vec{a} = \frac{d}{dt} \vec{p} = \vec{v} dt$$

$$= \left[\frac{1}{2} m v^2 \right]_{t_0}^{t_1}$$

$$= \boxed{\frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2}$$

Change in K.E.

$$W = \Delta KE = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2$$

Equate 1 & 2.

$$U(P_0) - U(P_1) = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2$$

$$U(P_0) + \frac{1}{2} m v_0^2 = U(P_1) + \frac{1}{2} m v_f^2$$

Conservation of Energy

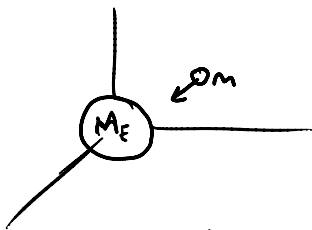
$$\Delta PE = \Delta KE$$

$$E_i = E_f$$

1013

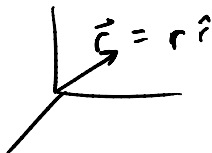
TENTHTEEN PRODUCTIONS

Escape Velocity from Earth



$$\vec{F} = - \frac{G M_E m}{r^2} \hat{r}$$

conservative force
Check $\nabla \times \vec{F} = 0$



Energy is conserved

$$E = \frac{1}{2} m v_i^2 + U_E = \frac{1}{2} m v_f^2 + U_{f \text{ at } \infty}$$

initial velocity gravitational potential energy final speed at $\infty = 0$

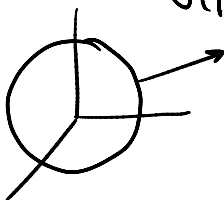
Minimum velocity you can have is 0 at ∞

$$\int_{P_0}^{P_1} \vec{F} \cdot d\vec{r} = \int_{P_0}^{P_1} -\nabla U \cdot d\vec{r} = -U \Big|_{P_0}^{P_1} = U(P_0) - U(P_1)$$

Choose P_0 to be a pt on Earth's surface. $r = R_E = \text{radius of earth}$

$P_1 = \text{point at infinity } (r \rightarrow \infty)$

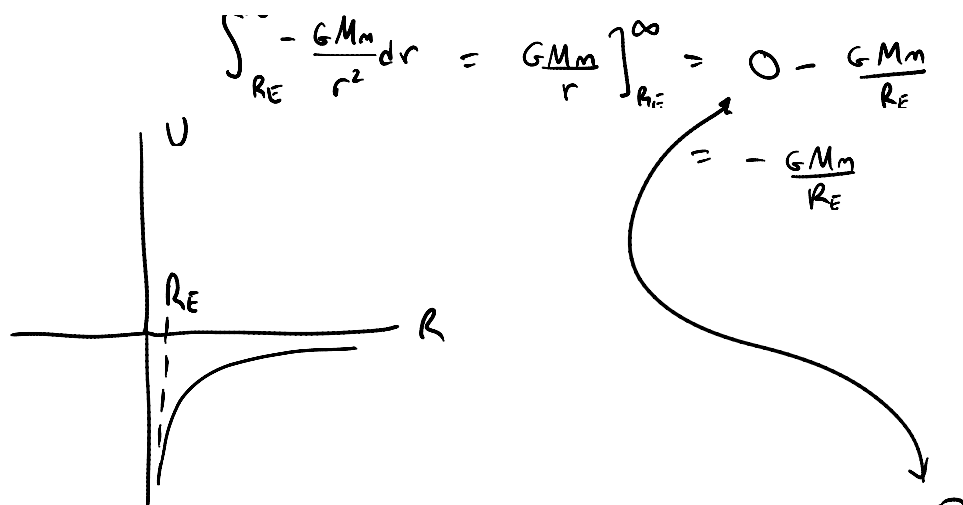
$$U(P_0) - U(P_1) = U_E - U_f = U(R) - U(\infty)$$



$$\vec{F} \cdot \frac{d\vec{r}}{dr} = - \frac{G M m}{r^2} \hat{r} \cdot \frac{dr}{dr} \hat{r}$$

$$= - \frac{G M m}{r^2} dr$$

$$\int_{\infty}^R - \frac{G M m}{r^2} dr = G M m \Big|_{\infty}^R = \infty - G M m$$



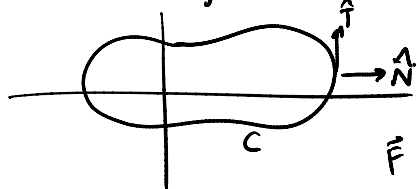
$$\frac{1}{2} m v_i^2 - \frac{GM_E m}{R_E} = \frac{1}{2} m v_f^2 + U_f$$

$$\frac{1}{2} m v_i^2 = \frac{GM_E m}{R_E}$$

$$v_i^2 = 2 \frac{GM_E}{R_E}$$

$$\text{escape velocity} = v_i = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2(6.6 \times 10^{-11})(M_E)}{R_E}} \approx 11 \text{ km/s}$$

Line integrals in 2D



$$\oint_C \vec{F} \cdot \hat{T} ds = \text{work done by } \vec{F}$$

↑
unit tangent vector

$$\oint_C \vec{F} \cdot \hat{N} ds = \text{flux of } \vec{F} \text{ out of the curve } C$$



parametrized by t

$$\vec{r} = \langle x(t), y(t) \rangle$$

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

$$\hat{N} = \hat{T} \times \hat{k}$$

Line Integrals, Divergence

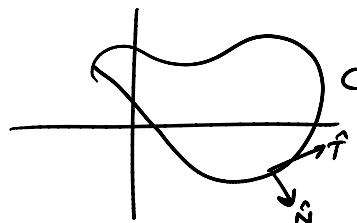
Friday, February 16, 2007
10:05 AM

Line Integrals in 2D

$$\oint_C \vec{F} \cdot \hat{T} ds$$

elmt. of arclength

$$\oint_C \vec{V} \cdot \hat{N} ds$$



$$C: \vec{r}(t) = \langle x(t), y(t) \rangle \quad t_0 \leq t \leq t_1$$

$$\vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$\hat{T} = \frac{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} \leftarrow \frac{ds}{dt} = \text{speed}$$

$$= \frac{dt}{ds} \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle = \hat{T}$$

$$\hat{T} ds = \langle dx, dy \rangle$$

$$\oint_C \vec{F} \cdot \langle dx, dy \rangle = \oint_C F_1 dx + F_2 dy = \oint_C \vec{F} \cdot \langle F_1(x,y), F_2(x,y) \rangle$$

$$= \oint_C \vec{F} \cdot d\vec{r} \quad \text{work done by } \vec{F} \text{ around } C$$

$$\hat{N} = \hat{T} \times \hat{k} = \begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j}$$

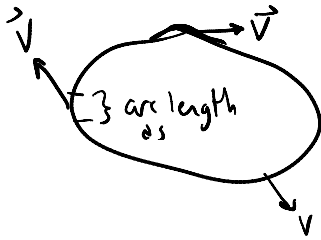
$$\hat{N} = \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle$$

$$\hat{N} ds = \langle dy, -dx \rangle$$

$$\text{Flux of } \vec{V} \text{ out of } C = \oint_C \vec{V} \cdot \langle dy, -dx \rangle = V_1 dy - V_2 dx \quad \vec{V} \text{ is not velocity}$$

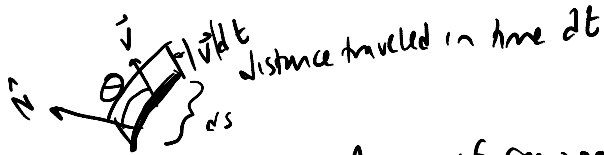
$\vec{V}$ is just some vector field

Interpret $\oint_C \vec{V} \cdot \hat{N} ds$ as a flux of $\vec{V}$ out of C



$\vec{v}$ = velocity vector field

Flux is equal to Area leaving C/unit time



Area of region is $(|\vec{v}|dt)(ds)\cos\theta |\hat{n}|$
 $= \vec{v} \cdot \hat{n} dt ds$

Flux = $\frac{A}{dt} = \vec{v} \cdot \hat{n} ds$

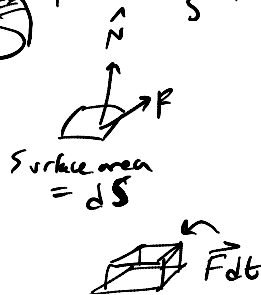
Total Flux = $\oint \vec{v} \cdot \hat{n} ds$

3D velocity field in some surface, Volume of fluid crossing surface in unit time.
 Flux of a vector field $\vec{F}$ through surface S is



$\iint_S \vec{F} \cdot \hat{n} dS$

$\oint$ closed surface



$z = \phi(x, y)$
 $z - \phi(x, y) = 0$

$\nabla(z - \phi)$
 $\hat{n} = \frac{\langle -\phi_x, -\phi_y, 1 \rangle}{\sqrt{\phi_x^2 + \phi_y^2 + 1}}$
 $dS = \sqrt{\phi_x^2 + \phi_y^2 + 1}$

Normal vector on sphere points in $\hat{r}$ direction.

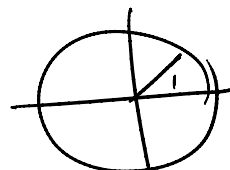
Sphere, radius a , $\hat{n} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$

normalize
 $\frac{\langle -\phi_x, -\phi_y, 1 \rangle}{\sqrt{\phi_x^2 + \phi_y^2 + 1}}$
 $\langle -\phi_x, -\phi_y, 1 \rangle$

$\frac{\langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle}{a \sin\phi}$

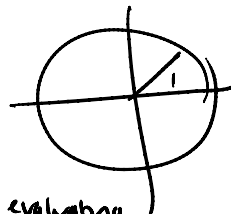
ex. Compute the flux of $\vec{F} = \langle x, y \rangle$ out of the unit circle
 $x^2 + y^2 = 1$

$\oint_C \vec{F} \cdot \hat{n} dS = 2\pi$



$$x^2 + y^2 = 1$$

$$\oint_C \vec{F} \cdot \hat{N} \, ds = 2\pi$$



$\hat{F} \cdot \hat{N} = 1$ b/c we are evaluating vector field on unit circle & $\parallel$.

$$\hat{F} = \hat{N} =$$

$$x = 1(\cos t) \quad y = 1(\sin t)$$

$$0 \leq t \leq 1$$

$$\vec{F} = \langle \cos t, \sin t \rangle$$

$$\hat{N} \, ds = \langle dy, -dx \rangle$$

$$= \langle \cos t, \sin t \rangle dt$$

$$\int_0^{2\pi} dt = 2\pi$$

or

$$\oint_C ds, \text{ curve is a circle so has to be } 2\pi$$

$$\oint_C \vec{F} \cdot \hat{T} \, ds = \text{work}$$

$$0 = \text{work}$$

b/c $\vec{F} \cdot \hat{T}$ perpendicular, so 0.

Divergence

Divergence of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$, denoted $\nabla \cdot \vec{F}$ or $\text{div}(\vec{F})$ is the following scalar quantity $\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\nabla \phi$
gradient

$\nabla \times \vec{F}$
curl

$\nabla \cdot \vec{F}$
divergence

Divergence is the "flux per unit volume" of a field $\vec{F}$.

$$\nabla \cdot \vec{F} = \lim_{\substack{V \rightarrow 0 \\ \text{volume} \rightarrow 0}} \frac{1}{V} \oint_S \vec{F} \cdot \hat{N} \, ds$$



∇ round
point
shrinks to 0
has bounding
surface

per unit
volume

Flux Integral - surface integral.

$\iint_S \vec{F} \cdot \hat{n} dS$ = Flux of vector field $\vec{F}$ through open surface S in direction $\hat{n}$.



$\oiint_S \vec{F} \cdot \hat{n} dS$ = flux of $\vec{F}$ out of closed surface S .



eg $\vec{F} = \vec{v}$ = velocity

$\iint_S \vec{v} \cdot \hat{n} dS$ = Volume of material going through S per unit time.

units = $\frac{\text{distance}}{\text{time}} \times \text{area} = \frac{\text{volume}}{\text{time}}$

$\iint_S \rho \vec{v} \cdot \hat{n} dS$ = Mass crossing S per unit time.
density $\times$ volume/time = mass/time

ex1

Find the flux of $\vec{F} = \langle x^2, y^2, z \rangle$ ^{upwards} through the surface $z = 1 - x^2 - y^2$ for $z > 0$

$$z = \phi(x, y) = 1 - x^2 - y^2$$



Paraboloid, vertex at $z=1$

$$\vec{F} = \langle x^2, y^2, 1 - x^2 - y^2 \rangle$$

$$\hat{n} dS = \langle -\phi_x, -\phi_y, 1 \rangle \text{ but normalize}$$

pointing upwards,
take gradient.

$$\frac{\langle -\phi_x, -\phi_y, 1 \rangle}{\sqrt{\phi_x^2 + \phi_y^2 + 1}}$$

$\hat{n}$ unit vector
normal to
 $z = \phi(x, y)$ pointing
upward.

$$\nabla(z - \phi) \perp \text{surface}, \hat{n}.$$

$$dS = \sqrt{1 + \phi_x^2 + \phi_y^2} dx dy \quad \text{surface integral.}$$

So:

$$\frac{\langle -\phi_x, -\phi_y, 1 \rangle}{\sqrt{1 + \phi_x^2 + \phi_y^2}} \sqrt{1 + \phi_x^2 + \phi_y^2} dx dy$$

$$\langle -\phi_x, -\phi_y, 1 \rangle dS$$

$$\langle 2x, 2y, 1 \rangle dx dy$$

$$\iint_D \langle 2x, 2y, 1 \rangle \cdot \hat{n} dS$$

$$\iint_S \langle x^2, y^2, 1-x^2-y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle dx dy$$

$$\iint_R (2x^2 + 2y^2 + 1 - x^2 - y^2) dx dy$$

↑ unit circle. b/c projected on xy plane. $z=0$.

$$dx dy = r dr d\theta$$

$$\int_0^{2\pi} \int_0^1 2r^4 (\cos^2 \theta + \sin^2 \theta) + r - r^3 dr d\theta$$

$$\int_0^{2\pi} \cos^2 \theta d\theta = \int_0^{2\pi} \cos \theta (1 + \sin^2 \theta)$$

$$\int_0^{2\pi} \cos \theta - \cos \theta \sin^2 \theta$$

$$\left[\sin \theta - \frac{\sin^3 \theta}{3} \right]_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^2 \theta d\theta = \int_0^{2\pi} \sin \theta (1 - \cos^2 \theta)$$

$$= \int_0^{2\pi} \sin \theta - \sin \theta \cos^2 \theta$$

$$\left[-\cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{2\pi} = 0$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{3i\theta} = (e^{i\theta})^3 = (\cos \theta + i \sin \theta)^3$$

$$\cos(3\theta) + i \sin(3\theta) = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$$

$$\cos(3\theta) = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

$$\sin(3\theta) = -3 \sin^3 \theta + 3 \cos^2 \theta \sin \theta$$

$$\cos(3\theta) = \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$$

$$\sin(3\theta) = -\sin^3 \theta + 3(1 - \sin^2 \theta) \sin \theta$$

$$\cos(3\theta) = \cos^3 \theta + 3 \cos^3 \theta - 3 \cos \theta$$

$$\sin(3\theta) = -\sin^3 \theta - 3 \sin^3 \theta + 3 \sin \theta$$

$$\cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta$$

$$\sin(3\theta) = -4 \sin^3 \theta + 3 \sin \theta$$

$$\cos^3 \theta = \frac{1}{4} (\cos(3\theta) + 3 \cos \theta)$$

$$\sin^3 \theta = -\frac{1}{4} (\sin(3\theta) - 3 \sin \theta)$$

new identity

$$= 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 = \frac{\pi}{2} \quad \checkmark$$

Ex 2

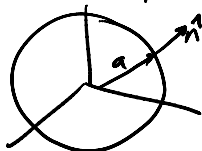
Calculate $\oint_S \vec{F} \cdot \hat{n} dS$ where S is the sphere $x^2 + y^2 + z^2 = a^2$

$$\vec{F} = \langle x, y, z \rangle$$

sphere, radius a



$$r = \langle x, y, z \rangle$$



$$\hat{n} = \frac{\langle x, y, z \rangle}{\sqrt{a^2}} = \frac{\langle x, y, z \rangle}{a}$$

$$\vec{r} \cdot \hat{n} = \frac{x^2 + y^2 + z^2}{a} = \frac{a^2}{a} = a \quad \text{Evaluate integrand on surface.}$$

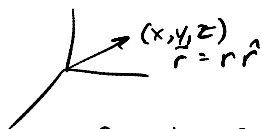
$$\oint \vec{r} \cdot \hat{n} dS = \oint a dS = a \oint dS = a \cdot 4\pi a^2$$

surface area $dS = 4\pi a^2$
So no integral necessary.

So you get factors of a as surface area grows, r radius extends.

Radial vector field; (choose so flux is same for every sphere around the origin.)

Suppose $\vec{F}$ is radially outward. For what functions $f(r)$ is the flux through a sphere independent of its radius?



Conjecture: $f = \frac{k}{r^2}$, k is a constant

$$\vec{F} = f(r) \hat{r}$$

$$\vec{F} \cdot \hat{n} dS = f(r) \hat{r} \cdot \hat{r} dS$$

$$\oint_S f(r) dS = f(a) \oint_S dS$$

$$\uparrow$$

$$x^2 + y^2 + z^2 = a^2$$

$$r = a$$

$$= f(a) 4\pi a^2$$

$$\underbrace{f(a)}_{\text{constant}} = 4\pi k$$

$$f(a) = \frac{k}{a^2}$$

$$f(r) = \frac{k}{r^2}$$

Electric field as a point charge $\rightarrow$

One good example.

Gravitational force also decays as $\frac{1}{r^2}$

Ex

Fluid w/ velocity $\vec{v} = (1 - x^2 - y^2) \hat{k}$ flowing in pipe $x^2 + y^2 = 1$
pipe is cylinder on z axis

Show that the flux through any surface $z = g(x, y)$ is the same.



decays to 0 at boundaries. have exact same flux of fluid through it no matter what surface.

Compact vector identity notation

Tuesday, February 20, 2007
10:01 AM

Compact notation for proving vector identities

$$\nabla \phi = \sum_{n=1}^3 \hat{i}_n \frac{\partial \phi}{\partial x_n}$$

divergence $\nabla \cdot \vec{F} = \sum_{n=1}^3 \hat{i}_n \cdot \frac{\partial \vec{F}}{\partial x_n}$

curl $\nabla \times \vec{F} = \sum_{n=1}^3 \hat{i}_n \times \frac{\partial \vec{F}}{\partial x_n}$

They all look really similar

in summary

$$\nabla * () = \sum_{n=1}^3 \hat{i}_n * \frac{\partial}{\partial x_n} ()$$

$$\hat{i}_1 = \hat{i} \quad x_1 = x$$

$$\hat{i}_2 = \hat{j} \quad x_2 = y$$

$$\hat{i}_3 = \hat{k} \quad x_3 = z$$

$$\nabla \phi, * = -, () = \phi = \text{scalar field}$$

$$\nabla \cdot \vec{F}, * = \cdot, () = \vec{F} = \text{vector field}$$

$$\nabla \times \vec{F}, * = \times, () = \vec{F} = \text{vector field}$$

Sometimes seasoned math veterans don't work Σ . It's Einstein's Notation, actually, it was developed by his secretary who noticed there was a Σ every time there was a subscript.

Check:

$$\begin{aligned} \nabla \phi &= \sum_{n=1}^3 \hat{i}_n \frac{\partial \phi}{\partial x_n} = \hat{i}_1 \frac{\partial \phi}{\partial x_1} + \hat{i}_2 \frac{\partial \phi}{\partial x_2} + \hat{i}_3 \frac{\partial \phi}{\partial x_3} \\ &= \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \sum_{n=1}^3 \hat{i}_n \cdot \frac{\partial \vec{F}}{\partial x_n} = \hat{i}_1 \cdot \frac{\partial \vec{F}}{\partial x_1} + \hat{i}_2 \cdot \frac{\partial \vec{F}}{\partial x_2} + \hat{i}_3 \cdot \frac{\partial \vec{F}}{\partial x_3} \\ &= \hat{i} \cdot \frac{\partial \vec{F}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{F}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{F}}{\partial z} \end{aligned}$$

$$\begin{aligned} \vec{F} &= \langle F_1, F_2, F_3 \rangle \\ \frac{\partial \vec{F}}{\partial x} &= \left\langle \frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial x}, \frac{\partial F_3}{\partial x} \right\rangle \end{aligned}$$

$$= \frac{\partial F_1}{\partial x} \hat{i} + \frac{\partial F_2}{\partial x} \hat{j} + \frac{\partial F_3}{\partial x} \hat{k} \quad \checkmark$$

$$\nabla \times \vec{F} = \sum_{n=1}^3 \hat{i}_n \times \frac{\partial \vec{F}}{\partial x_n} = \hat{i}_1 \times \frac{\partial \vec{F}}{\partial x_1} + \hat{i}_2 \times \frac{\partial \vec{F}}{\partial x_2} + \hat{i}_3 \times \frac{\partial \vec{F}}{\partial x_3}$$

$$R: \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$$L: \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ \frac{\partial F_1}{\partial x} & \frac{\partial F_2}{\partial x} & \frac{\partial F_3}{\partial x} \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ \frac{\partial F_1}{\partial y} & \frac{\partial F_2}{\partial y} & \frac{\partial F_3}{\partial y} \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial z} & \frac{\partial F_3}{\partial z} \end{vmatrix}$$

$$\left[-\hat{j} \left(\frac{\partial F_3}{\partial x} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} \right) \right] + \left[\hat{i} \left(\frac{\partial F_3}{\partial y} \right) - \hat{k} \left(\frac{\partial F_1}{\partial y} \right) \right] + \left[\hat{i} \left(\frac{\partial F_2}{\partial z} \right) + \hat{j} \left(\frac{\partial F_1}{\partial z} \right) \right]$$

$$\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle = R \quad \checkmark$$

First set of vector identities

(i) $\nabla \cdot (\Phi \vec{F}) = \Phi \nabla \cdot \vec{F} + \nabla \Phi \cdot \vec{F}$ You can guess this.

(ii) $\nabla \times (\Phi \vec{F}) = \Phi \nabla \times \vec{F} + \nabla \Phi \times \vec{F}$

(iii) $\nabla \cdot (\nabla \times \vec{F}) = 0$

(iv) $\nabla \times (\nabla \Phi) = 0 \quad \checkmark$ we know this. we can check to see if a vector field is conservative this way.

Proof of (i)

$$\nabla \cdot (\Phi \vec{F}) = \sum_{n=1}^3 \hat{i}_n \cdot \frac{\partial}{\partial x_n} (\Phi \vec{F})$$

product rule

$$\frac{\partial \Phi}{\partial x_n} \vec{F} + \Phi \frac{\partial \vec{F}}{\partial x_n}$$

$$= \sum_{n=1}^3 \hat{i}_n \cdot \left(\frac{\partial \Phi}{\partial x_n} \vec{F} \right) + \sum_{n=1}^3 \hat{i}_n \cdot \left(\Phi \frac{\partial \vec{F}}{\partial x_n} \right)$$

$$= \underbrace{\sum_{n=1}^3 \frac{\partial \Phi}{\partial x_n} \hat{i}_n \cdot \vec{F}}_{\nabla \Phi \cdot \vec{F}} + \Phi \underbrace{\sum_{n=1}^3 \hat{i}_n \cdot \frac{\partial \vec{F}}{\partial x_n}}_{\nabla \cdot \vec{F}}$$

$$= \nabla \Phi \cdot \vec{F} + \Phi \nabla \cdot \vec{F}$$

Proof of (ii)

is identical after replacing ' with x.

Proof of (iii)

$$\nabla \cdot (\nabla \times \vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

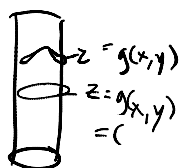
$$\frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Should all cancel
 $= 0$

Problem from yesterday
 Fluid velocity field

$$\vec{v} = \hat{k}(1 - x^2 - y^2)$$

Inside a pipe $x^2 + y^2 = 1$
 Show that flux of $\vec{v}$ through surface $z = g(x, y)$
 S is same for all functions g .
 $z = g(x, y)$



$$\iint_S \vec{v} \cdot \hat{n} \, dS$$

$$\langle 0, 0, 1 - x^2 - y^2 \rangle \cdot \langle 0, 0, 1 \rangle \, dx \, dy$$

$$\iint (1 - x^2 - y^2) \, dx \, dy \leftarrow \text{independent of } g \text{ after dot product}$$

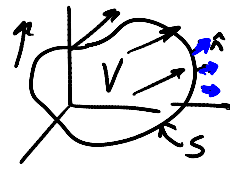
Same amt of fluid through any spot
 b/c you can't gain/lose fluid. No sources
 of fluid. This is why flux is same.

Divergence Theorem

Friday, February 23, 2007
10:02 AM

Divergence Theorem

Suppose a volume V is bounded by a surface S & there is a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$



$$\oint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

flux of $\vec{F}$ out of S .

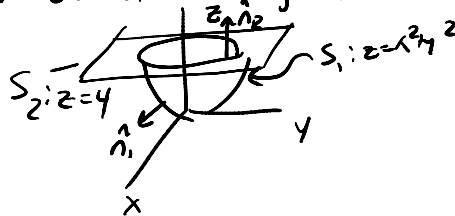
$\hat{n}$ is an outward normal.

The divergence of a vector field is a tiny sphere around each pt & the divergence tells you a flux density of that sphere (how much fluid/time is in the sphere)

$$\nabla \cdot \vec{F} = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \oint_{\Delta S} \vec{F} \cdot \hat{n} dS$$

Ex

Find the flux of $\vec{F} = x\hat{i} + y\hat{j} + 3\hat{k}$ out of the region bounded by $\vec{r} = x^2 + y^2$ & $z = 4$



$$\oint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

↑ direct method ↑ method 2.

$$\oint_S \vec{F} \cdot \hat{n} dS = \iint_{S_1} \vec{F} \cdot \hat{n}_1 dS + \iint_{S_2} \vec{F} \cdot \hat{n}_2 dS$$

$$\hat{n}_1 dS = \langle 2x, 2y, -1 \rangle dx dy$$

↑ negative of the normal normal b/c it's pointing outward

$$\langle x, y, 3 \rangle \cdot \langle 0, 0, 1 \rangle dx dy$$

$$\langle x, y, 3 \rangle \cdot \langle 2x, 2y, -1 \rangle dx dy$$

$$(2x^2 + 2y^2 - 3) dx dy$$

Integrating over circle w/ radius 2.

$$\iint_R (2x^2 + 2y^2 - 3) dx dy + \iint_R 3 dx dy$$

1. 1. 1. 2π(2) = 12

So... no $3dx dy$ terms any more

has to be $3\pi(4) = 12\pi$ but they will cancel b/c of $-3dx dy$

$$2 \iint (x^2 + y^2) dx dy$$

$$2 \int_0^{2\pi} \int_0^2 r^3 dr d\theta$$

$$2 \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = 2 \int_0^{2\pi} 4 d\theta = 8(2\pi) = 16\pi$$

Divergence theorem

$$\iiint \nabla \cdot \vec{F} dV$$

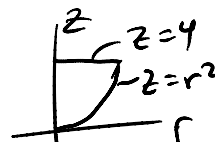
$$2 \int_0^{2\pi} \int_0^2 \int_0^4 r dz dr d\theta$$

$$2 \int_0^{2\pi} \int_0^2 zr \Big|_0^4 dr d\theta$$

$$\langle x, y, z \rangle$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 1 + 1 + 0 = 2$$



$$4\pi \int_0^2 4r - r^3 dr$$

$$4\pi \left(\frac{4r^2}{2} \Big|_0^2 - \frac{r^4}{4} \Big|_0^2 \right) = 16\pi$$

Notation

Laplacian (ϕ) = $\nabla^2 \phi = \nabla \cdot (\nabla \phi)$ divergence of gradient.
($= \Delta \phi$)

$$\nabla \phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

$$\nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Vector identities (set 2)

(v) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$

$$\nabla^2 \vec{F} = \langle \nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3 \rangle = \left\langle \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}, \right.$$

etc.

(vi) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

(vii) $\nabla \times (\vec{F} \times \vec{G}) = (\nabla \cdot \vec{G}) \vec{F} - (\nabla \cdot \vec{F}) \vec{G} + (\vec{F} \cdot \nabla) \vec{G} - (\vec{G} \cdot \nabla) \vec{F}$

$$\vec{G} \cdot \nabla = \langle G_1, G_2, G_3 \rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$= G_1 \frac{\partial}{\partial x} + G_2 \frac{\partial}{\partial y} + G_3 \frac{\partial}{\partial z}$$

$$(\vec{G} \cdot \nabla) \vec{F} = \left\langle G_1 \frac{\partial F_1}{\partial x} + G_2 \frac{\partial F_1}{\partial y} + G_3 \frac{\partial F_1}{\partial z}, G_1 \frac{\partial F_2}{\partial x} + G_2 \frac{\partial F_2}{\partial y} + G_3 \frac{\partial F_2}{\partial z}, \right.$$

etc.

(viii) $\nabla(\vec{F} \cdot \vec{G}) = \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) + (\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F}$

Application

Maxwell's equations for electromagnetism & electromagnetic waves.

Maxwell's equations for electromagnetism & electromagnetic waves.

In vacuum

$\nabla \cdot \vec{E} = 4\pi\rho$, ρ = charge density. $\vec{E}$ = electric field.
 Gauss' Law
 $\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ ← Faraday's Law

$\nabla \cdot \vec{B} = 0$ ← (No magnetic monopoles)

Ampere's Law
 $\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J}$ $\vec{J}$ = current density
 $\oint_S \vec{E} \cdot \hat{n} dS$



$= \iiint_V \nabla \cdot \vec{E} dV$

divergence
theorem

$\nabla \cdot \vec{E} = 4\pi\rho$

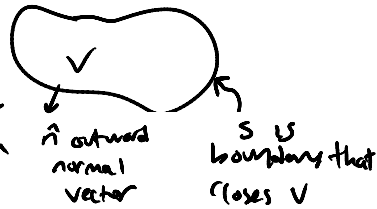
$4\pi \iiint_V \rho dV$

$= 4\pi Q$ ← total charge enclosed by S

Divergence Theorem + 2 corollaries to it

div theorem $\oint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV$ $\vec{F}$ is vector field

corollary 1: $\oint_S P \hat{n} dS = \iiint_V \nabla P dV$ P is scalar field, is a function.



Corollary 2: $\oint_S \hat{n} \times \vec{F} dS = \iiint_V \nabla \times \vec{F} dV$

In our compact vector notation,

$$\oint_S \hat{n} * () dS = \iiint_V \nabla * () dV$$

if $*$ = blank, we're talking about corollary 1, $()$ is scalar
 if $*$ = $\cdot$ or $\times$, it's corollary 2 & $()$ is vector.

Corollary 1 Use the divergence theorem to show $\oint_S P \hat{n} dS = \iiint_V \nabla P dV$
 Prove it component by component, $\hat{i} \cdot \oint_S P \hat{n} dS = \hat{i} \cdot \iiint_V \nabla P dV$

$$\hat{i} \cdot \oint_S P \hat{n} dS = \hat{i} \cdot \iiint_V \nabla P dV$$

$$\hat{k} \cdot \oint_S P \hat{n} dS = \hat{k} \cdot \iiint_V \nabla P dV$$

Show first one:

$$\begin{aligned} \hat{i} \cdot \oint_S P \hat{n} dS &= \oint_S P \hat{i} \cdot \hat{n} dS \\ &= \iiint_V \nabla \cdot P \hat{i} dV \end{aligned}$$

Now we can apply the divergence theorem

$$= \iiint_V \frac{\partial P}{\partial x} dV$$

↑ the $\hat{i}$ component of ∇P

$$\begin{aligned} &= \iiint_V \hat{i} \cdot \nabla P dV \\ &= \hat{i} \cdot \iiint_V \nabla P dV \end{aligned}$$

which is what we wanted.

Corollary 2

Idea: Apply divergence theorem to $\vec{F} \times \vec{a}$, where $\vec{a}$ is any constant vector.

$$\oint_S (\vec{F} \times \vec{a}) \cdot \hat{n} dS = \iiint_V \nabla \cdot (\vec{F} \times \vec{a}) dV \quad \text{by div. theorem}$$

you can think of this as a parallelepiped.
 $-\vec{x}$ gives area, $\cdot \hat{n}$ gives height

$$\nabla \cdot (\vec{F} \times \vec{a}) = \vec{a} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{a})$$

If $\vec{a}$ is constant vector field,
 then $\nabla \times \vec{a}$ is 0.



$$\nabla \cdot (\vec{F} \times \vec{a}) = \vec{a} \cdot (\nabla \times \vec{F})$$

$$(\vec{F} \times \vec{a}) \cdot \hat{n} = (\hat{n} \times \vec{F}) \cdot \vec{a}$$

Cyclic permutation

$$\oint_S (\vec{F} \times \vec{a}) \cdot \hat{n} dS = \oint_S \vec{a} \cdot (\hat{n} \times \vec{F}) dS = \vec{a} \cdot \oint_S \hat{n} \times \vec{F} dS$$

$$\iiint_V \nabla \cdot (\vec{F} \times \vec{a}) dV = \iiint_V \vec{a} \cdot (\nabla \times \vec{F}) dV = \vec{a} \cdot \iiint_V (\nabla \times \vec{F}) dV$$

b/c $\vec{a}$ is constant, can take it out.

$\vec{a}$ can be $\langle \hat{i}, \hat{j}, \hat{k} \rangle$, $\hat{i}$, $\hat{j}$, or $\hat{k}$

$$\vec{a} \cdot \oint_S \hat{n} \times \vec{F} dS = \vec{a} \cdot \iiint_V (\nabla \times \vec{F}) dV$$

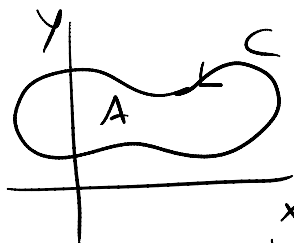
this is identical to this b/c $\vec{a} \cdot \underline{\quad} = \underline{\vec{a} \cdot \quad} = \underline{\quad}$

$$\text{So } \oint_S \hat{n} \times \vec{F} dS = \iiint_V (\nabla \times \vec{F}) dV \quad \text{since } \vec{a} \text{ is arbitrary.}$$

This proves 2nd corollary.

Green's Theorem

Line integrals in a plane, vector fields defined in 2 dimensions.



Area = region bounded by C

C must be counter-clockwise closed curve, single.

$$\oint_C M(x,y) dx + N(x,y) dy = \iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

goes from line integral to area integral.

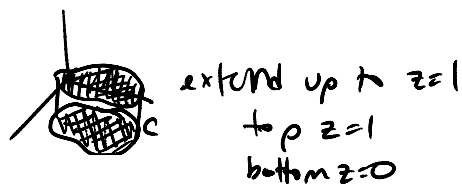
Use the divergence theorem to prove Green's theorem.

Use the divergence theorem to prove Green's theorem.

$$\oint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

$$\text{Let } \vec{F} = \langle N(x,y), -M(x,y), 0 \rangle \Rightarrow \nabla \cdot \vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$V =$



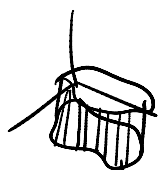
$$\oint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$



$$\iint_{\text{sides}} \vec{F} \cdot \hat{n} dS + \iint_{\text{top}} \vec{F} \cdot \hat{k} dS + \iint_{\text{bottom}} \vec{F} \cdot -\hat{k} dS$$

$$\begin{aligned} \vec{F} \cdot \hat{k} &= 0 \\ \vec{F} \cdot -\hat{k} &= 0 \end{aligned}$$

So only $\iint_{\text{sides}}$



height $\times$ arc length (s)

only need line integral to sum the areas.

$1 \times s$

$$\oint_C \vec{F} \cdot \hat{n} ds$$

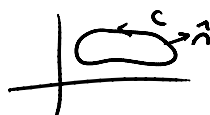
$$\iiint_V \nabla \cdot \vec{F} dV = \iiint_V \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dz dx dy$$

$$\iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \int_0^1 dz dx dy$$

Integral doesn't depend on z

$$\iint_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$= \oint_C \vec{F} \cdot \hat{n} ds$$



$$\begin{aligned} \hat{n} &= \hat{i} \times \hat{k} \\ \hat{i} ds &= d\vec{r} = \langle dx, dy \rangle \end{aligned}$$

$$= \oint_C \vec{F} \cdot (\hat{i} \times \hat{k}) ds$$

$$\begin{aligned} \vec{F} \cdot (\hat{i} \times \hat{k}) &= (\hat{i} \times \hat{k}) \cdot \vec{F} = (\hat{k} \times \vec{F}) \cdot \hat{i} \\ \oint_C (\hat{k} \times \vec{F}) \cdot \hat{i} ds \end{aligned}$$

2.5.2.2

$$\oint_C (\vec{F} \times \vec{F}) \cdot d\vec{r}$$

$$\vec{F} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ N-M & 0 & 0 \end{vmatrix}$$

$$\hat{i}(M) - \hat{j}(-N) \\ = \langle M, N, 0 \rangle$$

$$\oint_C Mdx + Ndy$$

Archimedes' Principle

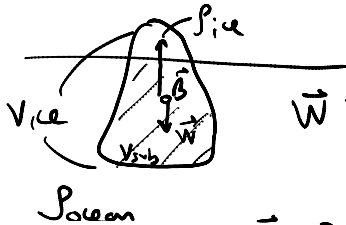
Tuesday, February 27, 2007

10:00 AM

Archimedes' Principle

When a body is totally or partially immersed in a liquid & in equilib, then the buoyancy force is equal to the weight of the displaced liquid. Acts at center of mass of displaced liquid.

example: floating ice berg (or ice cube)



$$\vec{W} = \text{weight of ice berg} = -M \vec{g} \hat{k} = -\rho_{ice} V_{ice} \vec{g} \hat{k}$$

$$\begin{aligned} \vec{B} + \vec{W} &= 0 \quad \text{since ice berg is at rest} \\ \vec{B} &= \rho_{ice} V_{ice} \vec{g} \hat{k} \quad \text{by Archimedes, } |\vec{B}| = \rho_{ocean} V_{sub} g \\ \vec{W}_{ice} &= \rho_{ice} V_{ice} \vec{g} = \rho_{ocean} V_{sub} \vec{g} \\ \frac{V_{sub}}{V_{ice}} &= \frac{\rho_{ice}}{\rho_{ocean}} \leftarrow \text{less dense than ocean. So } \frac{\rho_{ice}}{\rho_{ocean}} < 1. \end{aligned}$$

Density in the ocean varies (colder temps make denser areas)

Idea behind Archimedes

What is buoyancy force?

$\vec{B}$ = force exerted on surface of body due to fluid pressure.
Pressure is force/area



dS surrounding fluid exerts a force inward.

$\vec{F} = p dS (\hat{n})$ where p is pressure, $\hat{n}$ is outward normal

$$\text{Buoyancy force } \vec{B} = - \oint_S p \hat{n} dS$$

outward normal

Buoyancy force is the same if V is filled with the liquid or if V is filled with some other material.

if V is filled w/ the liquid & liquid is at rest then $|\vec{B}| = |\vec{W}_{\text{liquid}}|$
 or if V is filled w/ some other material & at rest then $|\vec{B}| = |\vec{W}_{\text{mat}}|$

pressure inside a fluid at rest

let V be ^{arbitrary} test volume inside the fluid



$$\vec{B} + \vec{W} = 0$$

$$\vec{B} = - \oint_S p \hat{n} dS$$

$$\vec{W} = - \vec{g} \hat{k} \iiint_V \rho dV$$

$$\vec{B} + \vec{W} = 0 = - \oint_S p \hat{n} dS - \vec{g} \hat{k} \iiint_V \rho dV$$

Trick to extract local pressure

Use corollary to divergence theorem

$$\oint_S p \hat{n} dS = \iiint_V \nabla p dV$$

$$- \iiint_V \nabla p dV - \vec{g} \hat{k} \iiint_V \rho dV = 0$$

$$\iiint_V (\nabla p + \rho \vec{g} \hat{k}) dV = 0$$

Since V is arbitrary, it must be that the integrand is 0

$$\nabla p = -\rho \vec{g} \hat{k}$$

eqn that $p(x, y, z)$ satisfies in a fluid that is at rest

note: $p(x, y, z)$ must satisfy $\nabla \times (\rho \vec{k}) = 0$

$$\nabla p = -\rho \vec{k}$$

$$\nabla \times \nabla p = -\rho (\nabla \times \vec{k})$$

$$0 = -\rho (\nabla \times \vec{k})$$

$$0 = \nabla \times \vec{k}$$

Take curl of both sides

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = 0$$

ex Model of ocean density:

$$\rho(z) = \rho_r - \alpha z$$

ρ_r $\nearrow$ constant, (+)
density at ocean surface
 $z=0$

$\rho(z)$ varies linearly w/
depth.

depth $= -z$

$p = p_0 = \text{atm. pressure } z=0 \quad \rho = \rho_r$

$\downarrow$

$p = p_r - \alpha z = p_r + \alpha |z|$

$$\nabla p = -\vec{g}(\rho_T - \alpha z)$$

$$\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\vec{g}(\rho_T - \alpha z)$$

$$\int_0^z \frac{\partial p}{\partial z} dz = \int_0^z -\vec{g}(\rho_T - \alpha z) dz$$

$$p(z) \Big|_0^z =$$

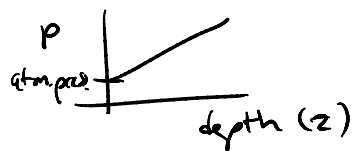
$$p(z) - p_0 = -\vec{g}(\rho_T z - \alpha \frac{z^2}{2})$$

$$p(z) = p_0 - \vec{g}(\rho_T z - \alpha \frac{z^2}{2})$$

$$p(z) = p_0 - \vec{g}\rho_T z + \vec{g}\alpha \frac{z^2}{2} \quad \text{quadratic}$$

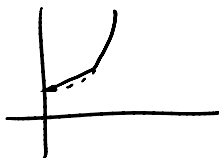
In swimming pool, not deep enough for fluid to vary.
So $\alpha = 0$

$$p(z) = p_0 - \vec{g}\rho_T z$$



In ocean, with $\alpha_0 = 0$

$$p = p_0 - \vec{g}\rho_T z + \vec{g}\alpha \frac{z^2}{2}$$



Green's Theorem, Work

Thursday, March 01, 2007
10:01 AM

Green's Theorem

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



As you go around the curve, region is on left

This makes C

Example

Show that area of R can be written

$P(x,y)$ $Q(x,y)$ pos.

$$A = \oint_C x dy \quad \text{where } C \text{ is the}$$

$$A = \iint_R dA \quad \text{so you want to choose } P \text{ or } Q \text{ such that}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$$Q = x, P = 0$$

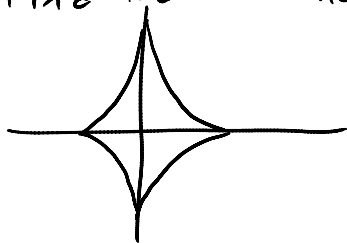
P or Q are same function of x or y

$$A = -\oint_C y dx$$

$$A = \frac{1}{2} \oint_C -y dx + x dy$$

Example

Find the area bounded by $y^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}}$



Envelope of a ladder falling

$$x = \cos^3 t$$

$$y = \sin^3 t$$

$$0 \leq t \leq 2\pi$$

$$dx = -3 \sin t \cos^2 t dt$$

$$dy = 3 \cos t \sin^2 t dt$$

$$A = \frac{3}{2} \int_0^{2\pi} \sin^4 t \cos^2 t + \cos^4 t \sin^2 t dt$$

$$= \frac{3}{2} \int_0^{2\pi} \sin^2 t \cos^2 t dt$$

$$= \frac{3}{8} \int_0^{2\pi} \sin^2 2t dt$$

$$\int_0^{2\pi} \sin^2 t dt$$

$$\int_0^{2\pi} \cos^2 t dt$$

Add them = $\int_0^{2\pi} 1 dt$



$$= \textcircled{2\pi}$$

$$\frac{50}{3} \left(\frac{1}{2} 2\pi \right)$$

Example

Show that the Moment of inertia about the y axis of a region R w/ constant density $= 1$ is

$$\frac{1}{3} \oint_C x^3 dy$$

$$Q = \frac{1}{3} x^3 \quad P = 0$$

$$= \iint_R x^2 dA$$

Work

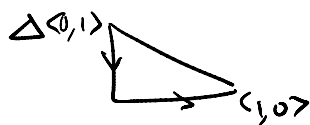
$$\vec{W} = \oint_C \vec{F} \cdot d\vec{r} = \oint_C F_1 dx + F_2 dy$$

$$= \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA \quad \vec{F} = \langle F_1, F_2, 0 \rangle$$

$$= \iint_R (\nabla \times \vec{F}) \cdot \hat{k} dA$$

Example

Find the work done to move a particle around perimeter of the $\Delta(0,1)$ by $\vec{F} = \langle 4y, -x^3 \rangle$

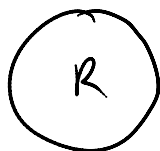


$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3x^2 - 3y^2$$

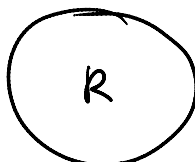
$$W = \iint_R (-3x^2 - 3y^2) dA$$

You can only apply Green's Theorem when P & Q have continuous partial derivatives

$\oint_C$ must be continuous & smooth, nonintersecting



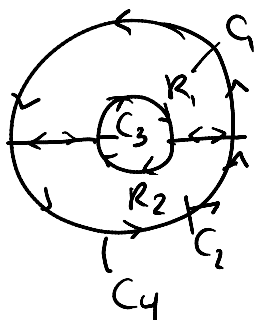
Not connected



Not simply connected



Simply connected



$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_{R_1} (\nabla \times \vec{F}) \cdot \hat{k} \, dA$$

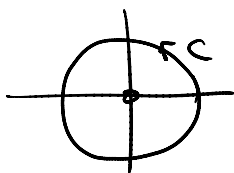
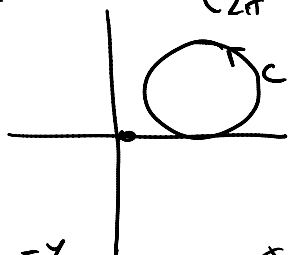
$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \iint_{R_2} (\nabla \times \vec{F}) \cdot \hat{k} \, dA$$

↔ cancel

$$\oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} = \iint_{R_1} (\nabla \times \vec{F}) \cdot \hat{k} \, dA - \iint_{R_2} (\nabla \times \vec{F}) \cdot \hat{k} \, dA = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} \, dA$$

Example

For $\vec{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ show that for a simple closed curve C

$$\oint_C \vec{F} \cdot d\vec{r} = \begin{cases} 0 & \text{if } C \text{ does not enclose the origin} \\ 2\pi & \text{if } C \text{ does enclose the origin} \end{cases}$$


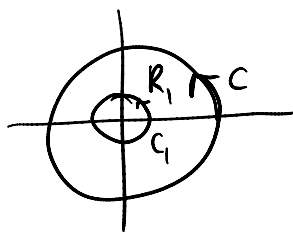
$$F_1 = -\frac{y}{x^2+y^2} \quad F_2 = \frac{x}{x^2+y^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \quad \text{So everywhere but the origin it's 0.}$$

But when you get to 0, it'll blow up. Can't divide by 0.

$$\frac{-0}{0^2+0^2} = ???$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \underbrace{\nabla \times \vec{F}}_0 \cdot \hat{k} \, dA = 0$$



Apply Green's Theorem to R_1

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_{R_1} (\nabla \times \vec{F}) \cdot \hat{k} \, dA = 0 \quad \text{b/c does not include origin}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r}$$

C_1 is something we draw. It's got radius a .

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_1} -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

$$= \int_0^{2\pi} -\frac{a \sin t}{a^2} dx + \frac{a \cos t}{a^2} dy$$

$$= \int_0^{2\pi} (-\sin t \sin t + \cos t \cos t) dt$$

$$\begin{aligned} x &= a \cos t \\ y &= a \sin t \\ 0 \leq t &\leq 2\pi \end{aligned}$$

$$= \int_0^{2\pi} \left(\frac{a^2 \sin t \sin t}{a^2} + \frac{a^2 \cos t \cos t}{a^2} \right) dt$$

$$= \int_0^{2\pi} dt = (2\pi)$$

Example

Suppose P & Q are P 's of x & y which have cts $\leftarrow$ continuous partial derivatives everywhere except $\langle 0, 0 \rangle, \langle 2, 0 \rangle, \text{ \& } \langle -3, 0 \rangle$

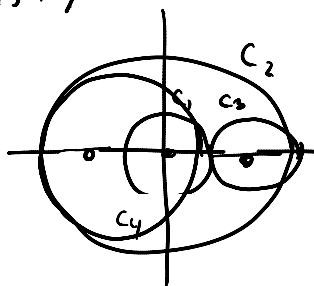
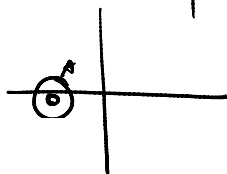
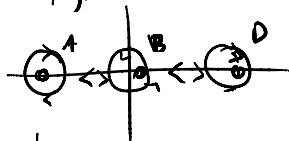
Suppose $\oint_C P dx + Q dy = 1 \quad C_1: x^2 + y^2 = 1$

$\oint_{C_2} P dx + Q dy = 5 \quad C_2: x^2 + y^2 = 9$

$\oint_{C_3} P dx + Q dy = 3 \quad C_3: (x-2)^2 + y^2 = 1$

What is $\oint_{C_4} P dx + Q dy \quad C_4: (x-1)^2 + y^2 = 4$

Integral around $C_2 =$
Integral around

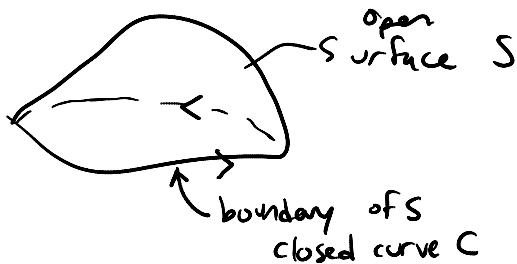


$$\oint_{C_4} = \int_A + \int_B + \int_D$$

Stoke's Theorem

Friday, March 02, 2007
10:04 AM

Stoke's Theorem relates to Green's Theorem



Consider a closed contour C & a surface S spanning C then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

related

where $\hat{n}$ is chosen in a right-handed sense.

Right hand rule



Green's Theorem is a special case of Stoke's Theorem b/c it is in x-y plane,

$$\hat{n} = \hat{k}$$



If you were walking around the region, it's always to your left

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

From Stoke's Theorem,

$R=S$ = flat surface

$$\vec{F} = \langle M(x,y), N(x,y), 0 \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

$$\oint \langle M(x,y), N(x,y), 0 \rangle \langle dx, dy, dz \rangle = \iint (\nabla \times \vec{F}) \cdot \hat{k} dx dy$$

b/c flat surface

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \hat{i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) + \hat{j} \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) + \hat{k} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

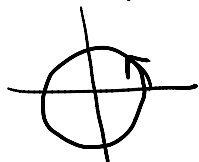
1M N 0'

Green's Theorem in vector form

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} \, dx \, dy \quad \text{b/c it's the case where curve is in } xy \text{ plane.}$$

Ex Compute $\oint_C \vec{F} \cdot d\vec{r}$ for $C: x^2 + y^2 = 1$ $\vec{F} = \langle 2x, -3x, 4y \rangle$
 Directly $\oint_C \vec{F} \cdot d\vec{r}$ or by Stoke's Theorem.

- 1) $S: x^2 + y^2 \leq 1, z=0$
 2) $S: z = 1 - x^2 - y^2, z \geq 0$

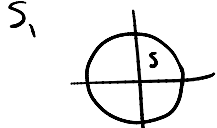


$x = \cos t$
 $y = \sin t \quad 0 \leq t \leq 2\pi$
 $z = 0$

$dx = -\sin t \, dt$
 $dy = \cos t \, dt$
 $dz = 0$

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \langle 2\cos t, -3\cos t, 4\sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt \\ &= \int_0^{2\pi} -3\cos^2 t \, dt \\ &= -3\pi \end{aligned}$$

$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n} \, dS$



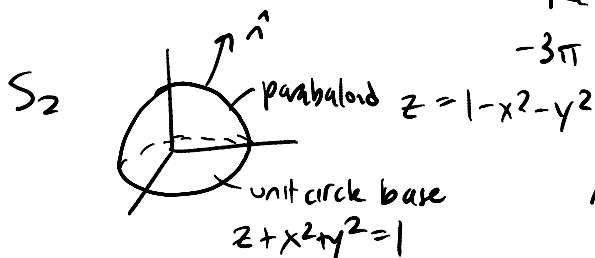
$\hat{k} = \hat{n}$ again

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2z & -3x & 4y \end{vmatrix} = \hat{i}(4) - \hat{j}(-2) + \hat{k}(-3) = \langle 4, -2, -3 \rangle$$

$$\iint_R \langle 4, -2, -3 \rangle \cdot \langle 0, 0, 1 \rangle \, dx \, dy$$

$$\iint_R -3 \, dx \, dy$$

Region, area of unit circle is π
 $-3\pi \checkmark$



$\langle -\phi_x, -\phi_y, 1 \rangle$
 $\hat{n} \, dS = \langle 2x, 2y, 1 \rangle$
 also gradient of

circle $\rightarrow \iint_R \langle 4, -2, -3 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx \, dy$

$$\begin{aligned} \iint_R (8x - 4y - 3) \, dx \, dy &= \int_0^{2\pi} \int_0^1 (8\cos\theta - 4\sin\theta - 3) \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \left(4\cos\theta - 2\sin\theta - \frac{3}{2} \right) \, d\theta \end{aligned}$$

FRIDAY!!!

$$\int_0^{2\pi} \int_0^{2\pi} (8r^2 \cos \theta - 4r^2 \sin \theta - 3r) dr d\theta$$

$$\int_0^{2\pi} \left[\frac{8r^3}{3} \cos \theta - \frac{4r^3}{3} \sin \theta - \frac{3r^2}{2} \right]_0^{2\pi} d\theta$$

$$\int_0^{2\pi} \left(\frac{8}{3} \cos \theta - \frac{4}{3} \sin \theta - \frac{3}{2} \right) d\theta$$

will be 0 b/c $-\sin \theta \Big|_0^{2\pi} = 0$, $-\cos \theta \Big|_0^{2\pi} = 0$
 $-\frac{3}{2} \theta \Big|_0^{2\pi} = -3\pi \checkmark$

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}_1 dS = \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n}_2 dS$$

provided S_1 & S_2 are both bounded by the same contour C . $\hat{n}_1$ & $\hat{n}_2$ are both chosen in the same right hand sense.



we can construct a closed volume w/ the 2 S 's.

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}_1 dS_1 = - \iint_{S_2} (\nabla \times \vec{F}) \cdot (-\hat{n}_2)$$

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n}_1 dS_1 + \iint_{S_2} (\nabla \times \vec{F}) \cdot (-\hat{n}_2) dS_2 = 0$$

$$\oint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = 0$$

$\hat{n}_1$ if top part,
 $\hat{n}_2$ if bottom part

Divergence Theorem says:

$$\oint_S \vec{G} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{G} dV$$

$$\oint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = \iiint_V \nabla \cdot (\nabla \times \vec{F}) dV$$

and so we know that the flux is 0

Mass & Momentum eqns.

Conservation of fluids.

Application of div. theorem

$$\oint_S \vec{F} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{F} dV$$

$\rho(x,y,z,t)$ density of fluid
 $\vec{v}(x,y,z,t)$ velocity " " Fluid



$$\text{let } M(t) = \text{mass inside } V \\ = \int_V \rho(x,y,z,t) dV$$

Arbitrary volume of fluid
 $S = \text{virtual surface bounding } V$

$$\frac{dM}{dt} = \text{flux into } S = \frac{d}{dt} \int_V \rho(x,y,z,t) dV = \int_V \frac{\partial \rho}{\partial t} dV$$

$$\text{Flux of mass into } S = \oint_S \rho \vec{v} \cdot \hat{n} dS$$

neg.

$$\int_V \frac{\partial \rho}{\partial t} dV = - \oint_S \rho \vec{v} \cdot \hat{n} dS \\ = - \int_V \nabla \cdot (\rho \vec{v}) dV \quad \text{divergence theorem}$$

$$\int_V \frac{\partial \rho}{\partial t} dV = - \int_V \nabla \cdot (\rho \vec{v}) dV \quad \text{for any } V \text{ volume in a fluid.}$$

possible

Hence it must be that integrands are the same.

Mass
 Conserv.
 eqn.

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{v})$$

eqn for density. need eqn for $\vec{v}$.

Momentum conservation

Particle
 eqn $\rightarrow \frac{d\vec{p}}{dt} = \vec{F}$

let $\vec{P}(t) = \text{momentum inside } V$

$$\vec{P}(t) = \int_V \rho(x,y,z,t) \cdot \vec{v} dV$$

vector addition. Adding up contrs.
 from each packet of fluid.

$$\frac{d\vec{P}}{dt} = \frac{d}{dt} \int_V \rho(x,y,z,t) \vec{v} dV = \int_V \frac{\partial}{\partial t} \rho \vec{v} dV$$

= Momentum flux into S + Forces

pressure forces
body force (like gravity or charged fluid w/ electric field) $\vec{F} = \frac{\text{force}}{\text{mass}}$

Flux of momentum into S
mass flux $\oint_S \rho \vec{v} \cdot \hat{n} dS$

for gravity, $\vec{F} = -g\hat{k}$

Momentum flux $\oint_S \rho \vec{v} \vec{v} \cdot \hat{n} dS$
b/c into

Pressure force $-\oint_S p \hat{n} dS$
 $\hat{n}$ is pointing outward & we want into
 p is pressure

Body forces $\int_V \vec{F} \rho dV$

$$\int_V \frac{\partial}{\partial t} (\rho \vec{v}) dV = -\oint_S \rho \vec{v} \vec{v} \cdot \hat{n} dS - \oint_S p \hat{n} dS + \int_V \rho \vec{F} dV$$

$-\oint_S \vec{\nabla} p \cdot dV$ comp. to divergence theorem

This is a vector eqn. Consider component by component

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{F} = \langle F_1, F_2, F_3 \rangle$$

↑ component of eqn

$$\int_V \frac{\partial}{\partial t} (\rho v_1) dV + \oint_S v_1 \rho \vec{v} \cdot \hat{n} dS = \int_V \left(\frac{\partial p}{\partial x} + \rho F_1 \right) dV$$

$$\int_V \left[\frac{\partial}{\partial t} \rho v_1 + \nabla \cdot (v_1 \rho \vec{v}) \right] dV = \int_V \left(-\frac{\partial p}{\partial x} + \rho F_1 \right) dV$$

V is arbitrary, therefore integrands must be the same.

$$\frac{\partial}{\partial t} (\rho v_1) + \nabla \cdot (v_1 \rho \vec{v}) = -\frac{\partial p}{\partial x} + \rho F_1$$

$$\frac{\partial \rho}{\partial t} v_1 + \rho \frac{\partial v_1}{\partial t} + v_1 \nabla \cdot (\rho \vec{v}) + \nabla v_1 \cdot (\rho \vec{v}) = -\frac{\partial p}{\partial x} + \rho F_1$$

$$v_1 \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) + \rho \frac{\partial v_1}{\partial t} + \nabla v_1 \cdot (\rho \vec{v}) = -\frac{\partial p}{\partial x} + \rho F_1$$

mass conservation

$$\nabla v_1 \cdot \rho \vec{v} = \left\langle \frac{\partial v_1}{\partial x}, \frac{\partial v_1}{\partial y}, \frac{\partial v_1}{\partial z} \right\rangle \cdot \langle \rho v_1, \rho v_2, \rho v_3 \rangle$$

$$\rho \frac{\partial v_1}{\partial t} + (\rho \vec{v} \cdot \nabla) v_1$$

Momentum eqn:

$$\rho \frac{\partial \vec{v}}{\partial t} + (\rho \vec{v} \cdot \nabla) \vec{v} = -\nabla p + \rho \vec{F} \quad \text{Euler eqn}$$

Eqn of Fluid Mechanics

Mass cons. eqn $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

Mom. cons. eqn.

$\frac{d\vec{p}}{dt} = \vec{F}$

$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = -\nabla p + \rho \vec{F}$

$\vec{F}$ is in dir. of greatest change.

$(\vec{v} \cdot \nabla) \vec{v} = (\langle v_1, v_2, v_3 \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle) \langle v_1, v_2, v_3 \rangle$

$= \langle (v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + v_3 \frac{\partial v_1}{\partial z}), (v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + v_3 \frac{\partial v_2}{\partial z}), (v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_3}{\partial y} + v_3 \frac{\partial v_3}{\partial z}) \rangle$

$\vec{v} \cdot \nabla$

Incompressible fluid where density cannot change. ρ constant.

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

$\rho (\nabla \cdot \vec{v}) = 0$
 $\nabla \cdot \vec{v} = 0$

Incompressible fluid divergence is 0.

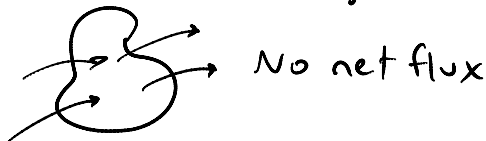
$\vec{v}$ field is divergence free.

if $\nabla \cdot \vec{v} = 0 \Rightarrow \oint_S \vec{v} \cdot \hat{n} dS = 0$

from divergence thm.

for any closed surface.

same amt fluid going in as going out.



No net flux

2-D fluid

could mean thin sheet of fluid or fluid in 3D only flowing in 2D, like parallel plates.

$\vec{v} = \langle v_1, v_2, 0 \rangle$

function of (x, y)

function of (x, y)

2-D incompressible fluid

2-D incompressible fluid

Described w/ stream function $\Psi(x, y)$

$$\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$$

or $\Psi(x, y, t)$

Scalar function Ψ

$$\left. \begin{aligned} v_1 &= \frac{\partial \Psi}{\partial y} \\ v_2 &= -\frac{\partial \Psi}{\partial x} \end{aligned} \right\} \nabla \cdot \vec{v} = \frac{\partial}{\partial x} \left(\frac{\partial \Psi}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial \Psi}{\partial x} \right) = 0$$

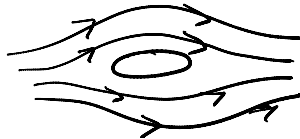
in the steady case where $\Psi(x, y) = \Psi$ & no time dependence, $\vec{v}$ doesn't depend on time.

$\Rightarrow$ "passive tracer" (dye particles) means dye moves on constant particles Ψ .

Streamline $\Psi(x, y) = C$ defines 1D curve in 2D space

$$\frac{d}{dt} (\Psi(x(t), y(t))) = \frac{\partial \Psi}{\partial x} \frac{dx}{dt} + \frac{\partial \Psi}{\partial y} \frac{dy}{dt} = \frac{\partial \Psi}{\partial x} v_1 + \frac{\partial \Psi}{\partial y} v_2 = 0$$

A particle dropped into fluid moves w/ fluid.



Heat Eqn

energy conservation eqn

energy in the form of thermal energy

Based on 2 hypotheses

- ① Heat required to raise temp of a volume of material ΔV by an amt ΔT is $\propto$ to ΔT & mass of V .

Heat to raise ΔV by $\Delta T \propto C \Delta T (\rho \Delta V)$

$$m C \Delta T$$

C = specific heat

- ② Heat flows from hot to cold in direction of ∇T & at a rate that depends on $|\nabla T|$

Heat flowing in time Δt across ΔS

∇ goes from cold to hot, opp from heat.

Heat flowing $= -k(\nabla T \cdot \hat{n}) \Delta S$ at outward heat flux

(if it was cold on inside

but hot on outside, would be $(+)$)



= Flux going outward

Heat required to raise V by ΔT in time $\Delta t =$

$$\iiint_V c_p \frac{\Delta T}{\Delta t} dV = \iint_S k(\nabla T \cdot \hat{n}) dS \frac{\Delta t}{\Delta t} \quad \text{inward heat flux}$$

$$\lim_{\Delta t \rightarrow 0} \iiint_V c_p \frac{\partial T}{\partial t} dV = \iint_S K(\nabla T \cdot \hat{n}) dS \quad K \text{ is coefficient of thermal conductivity}$$

$$= \iiint_V \nabla \cdot (K \nabla T) dV \quad \text{by div. theorem}$$



So integrands must be equal Since V is arbitrary

$$c_p \frac{\partial T}{\partial t} = \nabla \cdot (K \nabla T)$$



if c, ρ, k are constants, $\Rightarrow$

$$\boxed{\frac{\partial T}{\partial t} = K \nabla^2 T}$$

heat eqn,
 $K = \frac{k}{c_p} =$
 coefficient of
 thermometric
 conductivity