Functions, Derivatives, Limits

Tuesday, September 19, 2006
8:59 AM

http://www.esam.northwestern.edu/~silber

<u>Functions</u> - a rule that associates a <u>unique</u> # to every value of its argument. unique being there is only one

<u>Ex.</u> Take a # x, <u>square</u> it, add 1, then take square root.

$f(x) = \sqrt{x^2+1}$ Valid for all x

$\sqrt{x}$, $x \geq 0$

$f(x) = \arcsin(x) = \sin^{-1}(x)$ $\boxed{-1 \leq x \leq 1}$

sin is always limited to between -1 & 1

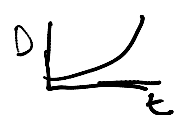
$|x| \leq 1$

<u>Ex.</u> Area A of a circle in terms of its radius R

$A(r) = \pi r^2$

<u>Ex.</u> Distance D of an object falling under gravity starting from rest, as a function of the elapsed time t.

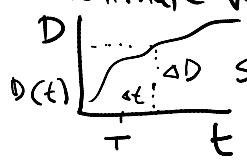
$D(t) = \frac{1}{2} g t^2$



D in meters
t in seconds
$g = 9.8 m/s^2$

<u>Motivating derivatives</u>

estimate velocity as $\bar{V} = \frac{\text{distance covered as time } \Delta t}{\text{interval of time } \Delta t}$



slope of straight line

$= \frac{D(T+\Delta t) - D(T)}{\Delta t}$

$\bar{V} = \frac{\Delta D}{\Delta t}$

instantaneous velocity

$V = \lim_{\Delta t \to 0} \frac{\Delta D}{\Delta t}$?

<u>Limits</u> essential to defining both derivative & integral

$\lim_{x \to a} f(x) = L$

qualitatively, as x gets closer & closer to a f(x) gets closer & closer to L

<u>ex.</u> $\lim_{x \to 0} \cos(x) = 1$

<u>ex.</u> $\lim_{x \to 1} \frac{x^3-1}{x-1} = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 1^2+1+1 = 3$

L'Hôpital's Rule

take derivative of top & bottom and it's the ratio.

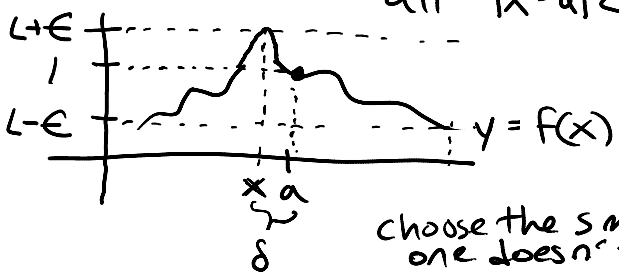
$$\frac{3x^2}{1} = \frac{3(1^2)}{1} = 3$$

$$\begin{array}{r} x^2+x+1 \\ x-1 \overline{\big) x^3-1} \\ -(x^3-x^2) \\ \hline x^2-1 \\ -(x^2-x) \\ \hline x-1 \\ -(x-1) \\ \hline 0 \end{array}$$

Do L'Hôpital's rule over & over again until you get a right answer. Sometimes, you'll <u>always</u> get $\frac{0}{0}$.

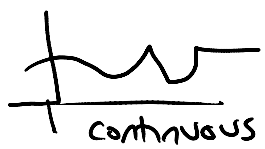
<u>Mathematical def of a limit</u>

$\lim_{x \to a} f(x) = L$ means that for all # $\epsilon > 0$ there exists a δ such that $|f(x) - L| < \epsilon$ for all $|x-a| < \delta$



choose the smaller distance b/c the larger one doesn't work - go ahead & try it.

<u>Continuous Function</u> - no jumps.



continuous

$f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$

discontinuous at $x=0$

<u>Mathematical definition of continuity</u> - A function $f(x)$ is continuous at $x=a$ if:

i) $\lim_{x \to a} f(x) = L$

ii) $f(a) = L$

Differentiation, Derivatives

Wednesday, September 20, 2006
8:56 AM

$$\int \frac{\sec^2 x}{1+\tan x}\,dx$$

¡Prrractico!

$u = 1+\tan x$

$du = \sec^2 x\,dx$

$$\int \frac{1}{u}\,du = \ln|u| + C$$
$$= \ln|1+\tan x| + C$$

Hyperbolic cosine of x = $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ $\quad i = \sqrt{-1}$

$e \approx 2.7\ldots$

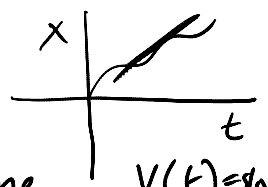
$e^{i\pi} = -1$

$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

$\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

<u>Derivatives</u>

velocity $\quad v(t) = \lim_{\Delta t}$



$x(t)$ = position as a function of time

$V(t)$ = slope of tangent line = time rate of change of position

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} = x'$$

NOTATION $= \dot{x}(t)$

<u>ex</u> $D(t) = \frac{1}{2}gt^2$

speed $= gt$

$$\frac{dD}{dt} = \lim_{\Delta t \to 0} \frac{D(t+\Delta t) - D(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\frac{1}{2}g(t-\Delta t)^2 - \frac{1}{2}gt^2}{\Delta t}$$

$$\frac{gt\Delta t + \frac{1}{2}g\Delta t^2}{\Delta t}$$

$$\lim_{\Delta t \to 0} gt + \frac{1}{2}g\Delta t = \boxed{gt} \text{ speed!}$$

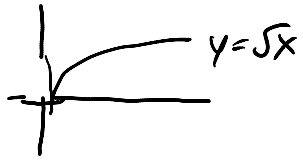
<u>ex</u> $f(x) = \sqrt{x}$

restriction $x > 0$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$\lim \sqrt{x+\Delta x} - \sqrt{x}$ ( ) $\Rightarrow$ $\lim \ldots$

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right) = \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}$$



$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Basic Rules of Differentiation

$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ Distributive-ish thing. :)

$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right)v + \left(\frac{dv}{dx}\right)u$ Product

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v[u'(x)] - u[v'(x)]}{v^2}$ Quotient

if $u = f(v)$ & $v = g(x)$

$u(x) = f(g(x))$

$\frac{du}{dx} = \frac{df}{dv}(g(x))\frac{dg}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}$ Chain Rule

ex. $y = \sqrt{1-x^2}$ $v = 1-x^2$, $y = \sqrt{v}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{v}} \cdot (-2x)$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

Multiple Derivatives

e.g. acceleration = time rate of change of velocity = time rate of change of position.

$$a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = x''(t) = \ddot{x}(t)$$

n^{th} derivative of $f(x)$ (n being an integer) notation $\frac{d^nf}{dx^n} = f^n(x)$

Standard Integration by Parts

Thursday, September 21, 2006
9:00 AM

1) $\int x^3 e^{-x^2}\,dx$ 2) $\int \ln(1+x)\,dx$

$u = -x^2$ $u = 1+x$

$du = -2x\,dx$ $du = dx$

$\frac{1}{2}\int u e^u\,du$ $\int \ln(u)\,du$

← STANDARD INTEGRATION BY PARTS →

Formula comes from the product rule.

$$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right)v + \left(\frac{dv}{dx}\right)u$$

$$uv = \int \frac{d}{dx}(uv)\,dx = \int\left(\frac{du}{dx}\right)v\,dx + \int\left(\frac{dv}{dx}\right)u\,dx$$ b/c $\int \frac{d}{dx} =$ function

$$= \int v\,du + \int u\,dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

1) $\frac{1}{2}\int y e^y\,dy$ LIPET

$u = y$ Log inverse functions polynomial exponent trig

$dv = e^y\,dy$ Kind of like PEMDAS.

$du = dy$ You want it to simplify the integral

$v = e^y$

$\int y e^y\,dy = y e^y - \int e^y\,dy = y e^y - e^y + C$ You can always check integration.

2) $\int \ln(y)\,dy$

$u = \ln(y)$ $du = \frac{1}{y}dy$

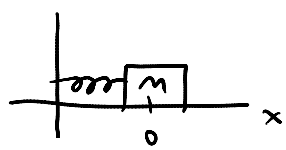
$dv = dy$ $v = y$

$\int \ln(y)\,dy = y\ln(y) - \int dy = y\ln(y) - y + C$

<u>Application</u> Newton's Law that force = mass · acceleration

$F = ma$

$a = \frac{d^2x}{dt^2}$ <u>ex</u>



$x=0$ is the rest position of the spring

Force due to spring is based on Hooke's Law → Force is proportionate to the distance

neglect friction

$F = -kx$ $k > 0$ ⇐ spring constant

↑ Force acts opposite to displacement

d^2x d^2x $-kx$ differential

$$m\frac{d^2x}{dt^2} = -kx \rightarrow \frac{d^2x}{dt^2} = \frac{-kx}{m}$$ differential equation for unknown function x

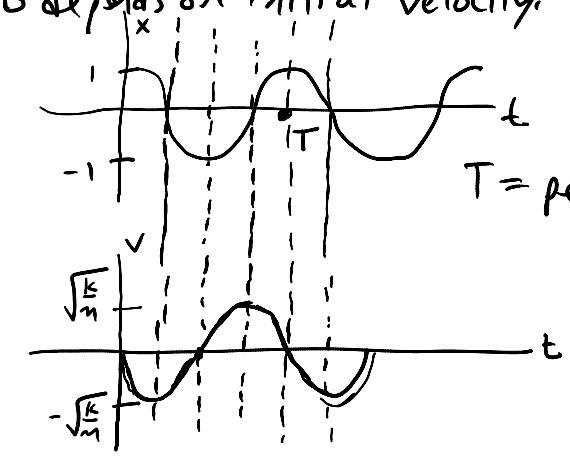
$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$v = x'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$
$$a = x''(t) = \underline{-A\omega^2\cos(\omega t)} - B\omega^2\sin(\omega t) = -\frac{k}{m}(A\cos(\omega t) + B\sin(\omega t))$$

$\omega^2 = \frac{k}{m}$, $\omega = \sqrt{\frac{k}{m}}$ and that's where the physics expression comes from.

$$x(t) = A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

A is the initial displacement
B depends on initial velocity.

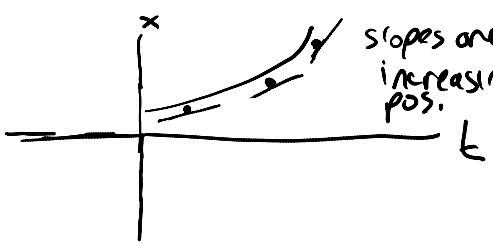
$A = 1$
$B = 0$
$x(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right)$
$v(t) = -\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)$



T = period of oscillation $= \frac{2\pi}{\omega}$ $\quad 2\pi = \sqrt{\frac{k}{m}}\,T$

$T = 2\pi\sqrt{\frac{m}{k}}$

As k increases, T decreases. Increase strength of spring, it doesn't have as much give and it doesn't oscillate as much.

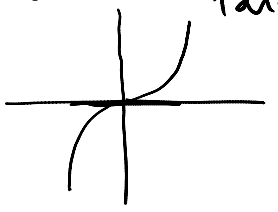


slopes are increasing and are pos.

$\frac{dx}{dt}$ = positive always > 0

$\frac{d^2x}{dt}$ = positive always (concave up) > 0

$\frac{dx}{dt} = 0$, horizontal tangent



not a max or min, which is unusual. it <u>is</u> an inflection point

Integral Practice

Thursday, September 21, 2006
10:53 AM

Miscellaneous Integrals

Instructions: For each of the following, you should be able to perform each of the integrations and differentiate the result to get back to the original integrand **without any errors**.

1. $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}\,dx$
2. $\displaystyle\int \frac{\sec^2 x}{1+\tan x}\,dx$
3. $\displaystyle\int \sin x \sec x\,dx$
4. $\displaystyle\int \frac{\csc x \cot x}{1+\csc^2 x}\,dx$
5. $\displaystyle\int \frac{\tan x}{\cos^2 x}\,dx$
6. $\displaystyle\int \csc^4 x\,dx$
7. $\displaystyle\int x\tan^2 x\,dx$
8. $\displaystyle\int x^2\cos^2 x\,dx$
9. $\displaystyle\int x^5\sqrt{2-x^3}\,dx$
10. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$
11. $\displaystyle\int \frac{x^2}{\sqrt{25+x^2}}\,dx$
12. $\displaystyle\int (\cos x)\sqrt{4-\sin^2 x}\,dx$
13. $\displaystyle\int \frac{1}{x^2-x+1}\,dx$
14. $\displaystyle\int \sqrt{x^2+x+1}\,dx$
15. $\displaystyle\int \frac{5x+31}{3x^2-4x+11}\,dx$
16. $\displaystyle\int \frac{x^4+1}{x^2+1}\,dx$
17. $\displaystyle\int \sqrt{x^4+x^7}\,dx$
18. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$
19. $\displaystyle\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$
20. $\displaystyle\int \frac{\cos 2x}{\cos x}\,dx$
21. $\displaystyle\int \frac{\tan x}{\ln(\cos x)}\,dx$
22. $\displaystyle\int \frac{x^7}{\sqrt{1-x^4}}\,dx$
23. $\displaystyle\int \ln(1+x)\,dx$
24. $\displaystyle\int x\sec^{-1} x\,dx$
25. $\displaystyle\int \sqrt{x^2+9}\,dx$
26. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
27. $\displaystyle\int \sqrt{2x-x^2}\,dx$
28. $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$
29. $\displaystyle\int \frac{x^4}{x^2-2}\,dx$
30. $\displaystyle\int \frac{\sec x\tan x}{\sec x+\sec^2 x}\,dx$
31. $\displaystyle\int \frac{x}{(x^2+2x+2)^2}\,dx$
32. $\displaystyle\int \frac{x^{1/3}}{x^{1/2}+x^{1/4}}\,dx$
33. $\displaystyle\int \frac{1}{1+\cos 2x}\,dx$
34. $\displaystyle\int \frac{\sec x}{\tan x}\,dx$
35. $\displaystyle\int \sec^3 x\tan^3 x\,dx$
36. $\displaystyle\int x^2\tan^{-1} x\,dx$
37. $\displaystyle\int x(\ln x)^3\,dx$
38. $\displaystyle\int \frac{1}{x\sqrt{1+x^2}}\,dx$
39. $\displaystyle\int e^x\sqrt{1+e^{2x}}\,dx$
40. $\displaystyle\int \frac{x}{\sqrt{4x-x^2}}\,dx$
41. $\displaystyle\int \frac{1}{x^3\sqrt{x^2-9}}\,dx$
42. $\displaystyle\int \frac{x}{(7x+1)^{17}}\,dx$
43. $\displaystyle\int \frac{4x^2+x+1}{4x^3+x}\,dx$
44. $\displaystyle\int \frac{4x^3-x+1}{x^3+1}\,dx$
45. $\displaystyle\int \tan^2 x\sec x\,dx$
46. $\displaystyle\int \frac{x^2+2x+2}{(x+1)^3}\,dx$
47. $\displaystyle\int \frac{x^4+2x+2}{x^5+x^4}\,dx$
48. $\displaystyle\int \frac{8x^2-4x+7}{(x^2+1)(4x+1)}\,dx$
49. $\displaystyle\int \frac{3x^5-x^4+2x^3-12x^2-2x+1}{(x^3-1)^2}\,dx$
50. $\displaystyle\int \frac{x}{x^4+4x^2+8}\,dx$
51. $\displaystyle\int (\ln x)^6\,dx$
52. $\displaystyle\int \frac{(1+x^{2/3})^{3/2}}{x^{1/3}}\,dx$
53. $\displaystyle\int \frac{\arcsin x)^2}{\sqrt{1-x^2}}\,dx$
54. $\displaystyle\int \frac{1}{x^{3/2}(1+x^{1/3})}\,dx$
55. $\displaystyle\int \tan^3 x\,dx$
56. $\displaystyle\int \sin^2 x\cos^4 x\,dx$
57. $\displaystyle\int \frac{xe^{x^2}}{1+e^{2x^2}}\,dx$

58. $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

59. $\displaystyle\int x^3 e^{-x^2}\,dx$

60. $\displaystyle\int \sin\sqrt{x}\,dx$

61. $\displaystyle\int \frac{\arcsin x}{x^2}\,dx$

62. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

63. $\displaystyle\int \sqrt{x^2-9}\,dx$

64. $\displaystyle\int x\sqrt{2x-x^2}\,dx$

65. $\displaystyle\int \frac{x-2}{4x^2+4x+1}\,dx$

66. $\displaystyle\int \frac{2x^2-5x-1}{x^3-2x^2-x+2}\,dx$

67. $\displaystyle\int \frac{e^{2x}}{e^{2x}-1}\,dx$

68. $\displaystyle\int \frac{\cos x}{\sin^2 x-3\sin x+2}\,dx$

69. $\displaystyle\int \frac{2x^3+3x^2+4}{(x+1)^4}\,dx$

70. $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+2\tan x+2}\,dx$

71. $\displaystyle\int \frac{x^3+x^2+2x+1}{x^4+2x^2+1}\,dx$

72. $\displaystyle\int \sin x\cos 3x\,dx$

73. $\displaystyle\int x^5\sqrt{x^3-1}\,dx$

74. $\displaystyle\int \ln(x^2+2x)\,dx$

75. $\displaystyle\int \frac{\sqrt{1+\sin x}}{\sec x}\,dx$

76. $\displaystyle\int \frac{1}{x^{2/3}(1+x^{2/3})}\,dx$

77. $\displaystyle\int \frac{\sin x}{\sin 2x}\,dx$

78. $\displaystyle\int \sqrt{1+\cos x}\,dx$

79. $\displaystyle\int \sqrt{1+\sin x}\,dx$

80. $\displaystyle\int \frac{\sec^2 x}{1-\tan^2 x}\,dx$

81. $\displaystyle\int \ln(x^2+x+1)\,dx$

82. $\displaystyle\int e^x\sin^{-1}(e^x)\,dx$

83. $\displaystyle\int \frac{\arctan x}{x^2}\,dx$

84. $\displaystyle\int \frac{x^2}{\sqrt{x^2-25}}\,dx$

85. $\displaystyle\int \frac{x^3}{(x^2+1)^2}\,dx$

86. $\displaystyle\int \frac{1}{x\sqrt{6x-x^2}}\,dx$

87. $\displaystyle\int \frac{3x+2}{(x^2+4)^{3/2}}\,dx$

88. $\displaystyle\int x^{3/2}\ln x\,dx$

89. $\displaystyle\int \frac{\sqrt{1+\sin^2 x}}{\sec x\csc x}\,dx$

90. $\displaystyle\int \frac{e^{\sqrt{\sin x}}}{\sec x\sqrt{\sin x}}\,dx$

91. $\displaystyle\int xe^x\sin x\,dx$

92. $\displaystyle\int x^2 e^{x^{3/2}}\,dx$

93. $\displaystyle\int \frac{\arctan x}{(x-1)^3}\,dx$

94. $\displaystyle\int \ln(1+\sqrt{x})\,dx$

95. $\displaystyle\int \frac{2x+3}{\sqrt{3+6x-9x^2}}\,dx$

96. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

97. $\displaystyle\int \frac{x^4}{(x-1)^2}\,dx$

98. $\displaystyle\int x^{3/2}\tan^{-1}(\sqrt{x})\,dx$

99. $\displaystyle\int \sec^{-1}(\sqrt{x})\,dx$

100. $\displaystyle\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

Partial Fns, Linear Approx

Friday, September 22, 2006
8:55 AM

$$\boxed{\sin^2 x + \cos^2 x = 1}$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\boxed{\cos^2 x = \tfrac{1}{2}(1+\cos(2x))}$$

$$\sin^2 x = \tfrac{1}{2}(1-\cos(2x))$$

$$\int \cos^2 x\,dx$$

$$\int \cos^3 x\,dx \quad \text{uses}$$
$$= \int \cos x \cdot \cos^2 x\,dx$$
$$= \int \cos x(1-\sin^2 x)\,dx$$
$$u = \sin x$$
$$du = \cos x\,dx$$
$$= \int \cos x\,(1-u^2)\,\frac{du}{\cos x}$$
$$= \int (1-u^2)\,du$$

$$\int \frac{\sec x}{\tan x}\,dx \qquad \int \sec^3 x \tan^3 x\,dx$$
$$= \int \frac{1}{\sin x}\,dx$$
$$= \int \frac{\sin x}{\sin^2 x}\,dx$$
$$= \int \frac{\sin x}{1-\cos^2 x}\,dx$$
$$\cos x = u$$
$$-\sin x\,dx = du$$
$$-\int \frac{1}{1-u^2}\,du$$

PARTIAL FRACTIONS

$$\frac{1}{1-u^2} = -\frac{1}{2}\left[\frac{1}{1+u} + \frac{1}{1-u}\right]$$
$$= -\frac{1}{2}\ln|1+u| + \frac{1}{2}\ln|1-u|$$

OR

$$\int \csc x\,dx$$
$$= \int \csc x\left(\frac{\cot x - \csc x}{\cot x - \csc x}\right)$$
$$= \int \frac{\csc x \cot x - \csc^2 x}{\cot x - \csc x}$$
$$u = \cot x - \csc x$$
$$du = -\csc^2 x + \csc x \cot x$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \ln|\cot x - \csc x| + C$$

$$\boxed{\text{PARTIAL FRACTIONS}} \rightarrow \text{Look That Up \& Practice}$$

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u}$$

$$\frac{A}{1+u} + \frac{B}{1-u}$$

$$\frac{1}{1-u^2} = \frac{A(1-u) + B(1+u)}{1-u^2}$$

$$A + B = 1 \qquad -A + B = 0$$

So they're both = to $\frac{1}{2}$

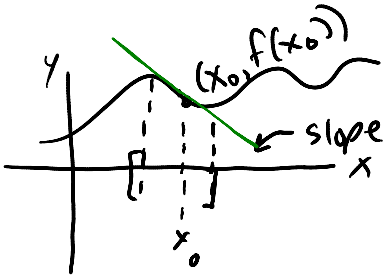
$$1 = A(1-u) + B(1+u)$$

if $u = 1$, $2B = 1$, $B = \frac{1}{2}$ if $u = -1$, $2A = 1$, $A = \frac{1}{2}$

Things to Know

Implicit Differentiation
Graphing
Differentiating inverse functions
Related Rates, Max/Mins

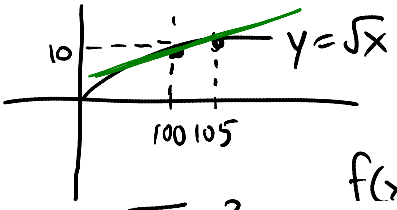
Linear Approximation – if you have a curve & you're just interested in a small range of x, then you can approximate it as a straight line.



$y - y_0 = m(x - x_0)$

x near x_0 $\quad y \approx f(x_0) + f'(x_0)(x - x_0)$

Approximate equation $y = f(x)$ near $(x, y) = (x_0, f(x_0))$ by a linear equation.



$\sqrt{105}$?

$x = 105$

$x_0 = 100$ taking tan at 100

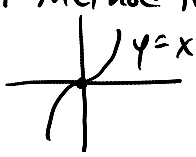
$f(x) = \sqrt{x}$ $\quad y_0 = 10$ $\quad f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$

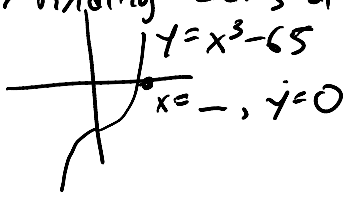
$y = 10 + \frac{1}{2n}(5)$

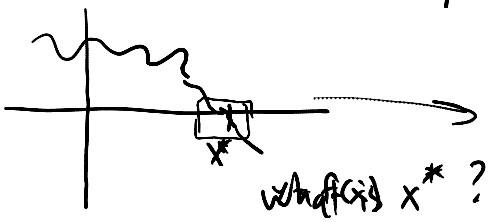
$y = 10 + \frac{1}{4} = 10.25$ $\qquad \sqrt{105} = 10.247$

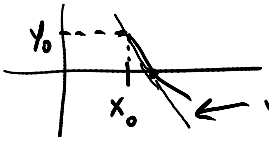
Newton – Raphson method for finding zeros of functions

$y = x^3 - 65$









$\leftarrow y = f(x_0) + f'(x_0)(x - x_0)$

what is x^* ?

$f(x^*) = 0$

x_0 = guess of x^*

$f(x_0) = 0$? No

That line crosses the x axis somewhere and that value is the closer value. Might just get you 0, or close enough

at $x = x_1$, $y = 0$

$= f(x_0) + f'(x_0)(x_1 - x_0)$

formula $\rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

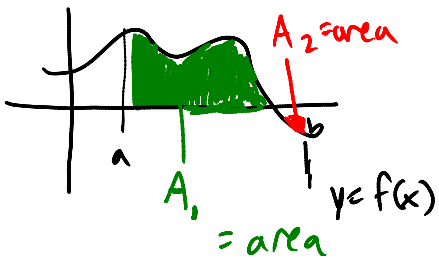
$\sqrt{105}$ let $y = f(x) = x^2 - 105$

Integration, Fundamental Thm of Calc

Monday, September 25, 2006
9:02 AM

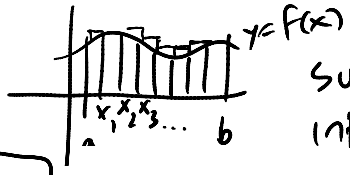
Integration

$\int_a^b f(x)\,dx =$ "signed area" under the curve $y=f(x)$ between $x=a$ & $x=b$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

Approximate integration.



Subdivide interval into n subintervals w/ $\Delta x = \frac{b-a}{n}$

Approximate area $\boxed{A \approx \sum_{i=1}^{n} f(x_i)\Delta x}$

Approximate area of ith subinterval $A_1 = wh = \Delta x\, f(x_i)$

more subintervals = better approximation.

$$A = \underbrace{\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x}_{\int_a^b f(x)\,dx}$$

And as $\Delta x \to 0$, $n \to \infty$

Connecting the antiderivative to the integral We want to show that integration and differentiation are inverse operations. Not obvious?

Fundamental Theorem of Calculus

(1) $\frac{d}{dx}\int_a^x f(u)\,du = f(x)$ ☆

2) $\int_a^x \frac{df}{dx}\,du = f(x) - f(a)$

or $\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$

$$\frac{d}{dx}\int_a^x f(u)\,du = \frac{d}{da}\left[\int_x^a f(u)\,du\right] = -f(a)$$

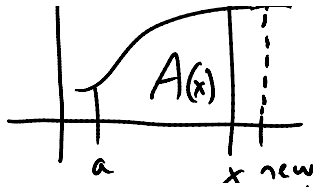
$$\boxed{\frac{d}{dx}\int_a^{g(x)} f(u)\,du = f(g(x))\cdot g'(x)}$$

let $v = g(x)$

$$\frac{d}{dx}\int_a^v f(u)\,du = \frac{d}{dv}\left[\int \quad\right]\cdot\frac{dv}{dx}$$

You're asking what is the rate of change of the integral.

let $A = \int_a^x f(u)\,du$



show $\frac{dA}{dx} = f(x)$ which is (1)

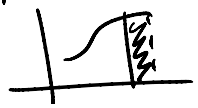
new value rate of change of integral.

$x + \Delta x$

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(u)\,du - \int_a^x f(u)\,du}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(u)\,du}{\Delta x}$$

Approximate the area of that space

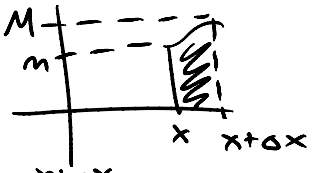


$$= \lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} = f(x)$$

b/c $lw = A$.

let $M = \max\{f(u) \text{ for } x < u < x + \Delta x\}$

let $m = \min\{f(u) \text{ for } x < u < x + \Delta x\}$



$$m\Delta x \le \int_x^{x+\Delta x} f(u)\,du \le M\Delta x$$

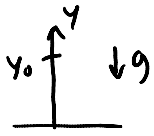
divide by Δx

$$m \le \frac{\int_x^{x+\Delta x} f(u)\,du}{\Delta x} \le M$$

m & M shrink together & become value of that point of the function. f(point)

Consider an object of mass m falling from a height $y = y_0$ at $t = t_0$ w/ initial velocity $v = v_0$ at $t = t_0$ w/ constant acceleration $a = -g$

Derive a formula for $y(t)$



$$y(t) = y_0 + v_0(t - t_0) - \tfrac{1}{2} g (t - t_0)^2$$

We just know this.

Mass is irrelevant b/c there's no dampening or anything.

$\frac{dv}{dt} = -g \longrightarrow \int \frac{dv}{dt}dt = -\int g\,dt$ or $\int_{v_0}^{v} \frac{dv}{dt'}dt' = -\int_{v_0}^{t} g\,dt'$

$v(t_0) = v_0$

$y(t_0) = y_0$

$\frac{dy}{dt} = v$

$v(t) = -gt + C$ ← combining integration constants.

$v(t_0) = -g t_0 + C$

$v_0 = -g t_0 + C$

$y(t_0) = y_0$

$\frac{dy}{dt} = v$

combining integration constants.

$$v(t_0) = -g t_0 + C$$
$$v_0 = -g t_0 + C$$
$$\underline{v_0 + g t_0} = C$$
$$= -g t + \underline{v_0 + g t_0} = v(t)$$ so simplify

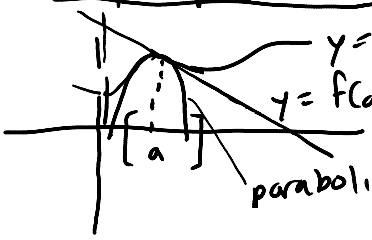
$$\int \frac{dy}{dt}\,dt = \int v_0 - g(t - t_0)\,dt$$

$$y(t) = v_0 t - \tfrac{1}{2} g (t - t_0)^2 + B$$
$$y(t_0) = y_0 = v_0 t_0 - \tfrac{1}{2}(0)^2 + B$$
$$y_0 - v_0 t_0 = B$$
$$y(t) = v_0 t - \tfrac{1}{2}(t - t_0)^2 + y_0 - v_0 t_0$$
$$y(t) = y_0 + v_0 (t - t_0) - \tfrac{1}{2} g (t - t_0)^2$$

Taylor Polynomials

Tuesday, September 26, 2006
8:55 AM

Taylor polynomials



$y = f(x)$

$y = f(a) + f'(a)(x-a)$

parabolic fit $y = C_0 + C_1(x-a) + C_2(x-a)^2$

choose coefficients, like linear approx.

Taylor polynomials approximate $y = f(x)$ for x close to $x=a$

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_n(x-a)^n$$

choose coefficients $C_0, C_1, \ldots C_n$ so that if you evaluate the polynomial at $x=a$, it will = function at $x=a$.

$P_n(a) = f(a)$ same point

$P_n'(a) = f'(a)$ same slope

$P_n''(a) = f''(a)$ same curvature

$P_n^{(n)}(a) = f^{(n)}a$

$(n+1)$ conditions on coefficients.

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \ldots + C_n(x-a)^n$$
$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \ldots + nC_n(x-a)^{n-1}$$
$$P_n''(x) = 2C_2 + 6C_3(x-a) + \ldots + n(n-1)C_n(x-a)^{n-2}$$
$$P_n^{(3)}(x) = 6C_3 + \ldots + n(n-1)(n-2)C_n(x-a)^{n-3}$$
$$\boxed{P_n^{(n)}(x) = n!\,C_n}$$

$$P_n(a) = C_0 = f(a)$$
$$P_n'(a) = C_1 = f'(a)$$
$$P_n''(a) = 2C_2 = f''(a)$$
$$P_n^{(3)}(a) = 3!\,C_3 = f^{(3)}(a)$$
$$P_n^{(n)}(a) = n!\,C_n = f^{(n)}(a)$$

$6 = 3! = 3 \cdot 2 \cdot 1$

$$\boxed{C_n = \frac{1}{n!} f^{(n)}(a)}$$

$$P_n(x) = \sum_{j=0}^{n} C_j (x-a)^j$$
$$= \sum_{j=0}^{n} \frac{1}{j!} f^{j}(a)(x-a)^j$$
$$= f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \ldots + \frac{1}{n!} f^{n}(a)(x-a)^n$$

There will be some error between your function & the polynomial.

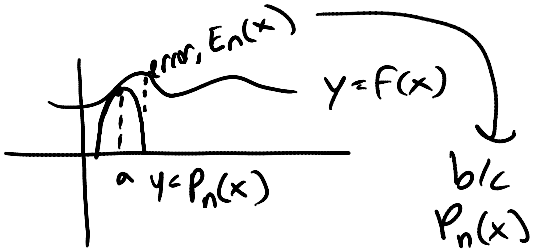
Taylor's Theorem

$f(x) = P_n(x) + E_n(x)$ → error term.

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \dots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

where the error $E_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(z)(x-a)^{n+1} + E_n(x)$

z is some # that we don't know. It's between x & a, though.



b/c $P_n(x)$ is diff from $f(x)$ right there. And in other places.

<u>Ex</u>. Use Taylor's Theorem to estimate $(1.3)^{\frac{1}{3}}$.

$a = \sqrt[3]{1}$

use $f(x) = x^{\frac{1}{3}}$ & expand about $x=a=1$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \dots + \underbrace{\frac{1}{3!}f^{(3)}(1)(x-1)^3}_{P_3(x)} + \underbrace{E_3(x)}_{\text{correction}}$$

& everything before

$f(x) = x^{\frac{1}{3}}$, $f(1) = 1$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, $f'(1) = \frac{1}{3}$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$, $f''(1) = -\frac{2}{9}$

$f^{(3)}(x) = \frac{10}{27}x^{-\frac{8}{3}}$, $f^{(3)}(1) = \frac{10}{27}$

$E_3(x) = \frac{1}{4!}f^{(4)}(z)(x-1)^4$ $\boxed{z \text{ is between } 1 \text{ \& } 1.3}$

$f^{(4)}(x) = -\frac{80}{81}x^{-\frac{11}{3}}$, $f^{(4)}(z) = -\frac{80}{81}z^{-\frac{11}{3}}$

$$P_n(x) = 1 + \frac{1}{3}(0.3) - \frac{1}{9}(0.3)^2 + \frac{5}{81}(0.3)^3 - \frac{1}{4!}\left(\frac{80}{81}\right)z(0.3)^4$$

where z is between 1 & 1.3

$z^{-\frac{11}{3}} \le 1$

$|E_3| < \frac{1}{4!}\left(\frac{80}{81}\right)(0.3)^4$

	$(1.3)^{\frac{1}{3}}$
1 term	
2 term	$1 + \frac{1}{3}(0.3) = 1.1$
3 terms	$1.1 - \frac{1}{9}(0.3)^2 = 1.09$
4 terms	$1.09 + \frac{5}{3^4}3^3 E{-3} = 1.09166666$
E_3	$\lvert E_3\rvert \le \frac{1}{4!}\left(\frac{80}{81}\right)(2^4)E^{-4} = .0003$

$4!(81)$

negative

Calculator says $(1.3)^{\wedge}(1/3) = 1.09139$

we got 1.09166

so error $\approx .00027$

Graphical, Algebraic Proofs

Wednesday, September 27, 2006
9:33 AM

$y = f(x)$ this defines the inverse function $x = f^{-1}(y) = g(y)$

$$f'(x) = \frac{1}{g'(y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

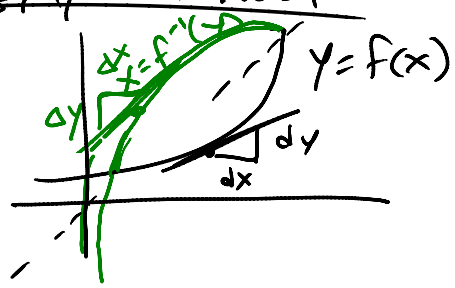
$$y = \tan^{-1}(x)$$

$$\frac{dy}{dx} = ?$$

$$x = \tan y$$

$$\frac{dx}{dy} = \sec^2 y \qquad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$$

Graphical Proof



now to get the slope you need $\frac{\Delta x}{\Delta y}$, which is why it's $\frac{dx}{dy}$

Algebraic Proof

$z = g(y)$

$y = f(x)$

$z = g(f(x))$

$$\frac{dz}{dx} = g'(\underbrace{f(x)}_{y}) \cdot \underbrace{f'(x)}_{\frac{dy}{dx}} = g'(y) \cdot \frac{dy}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$z = g(f(x)) = f^{-1}(f(x)) = x$$

$$\frac{dz}{dx} = 1$$

$$1 = g'(y) \cdot f'(x)$$

$$\Rightarrow f'(x) = \frac{1}{g'(y)}$$

HW

Staples

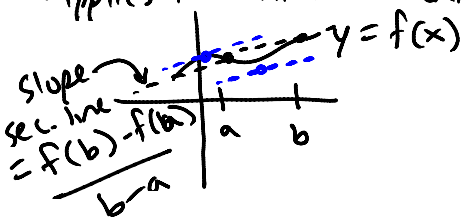
work all steps

Mean Value Thm

Thursday, September 28, 2006
9:04 AM

Mean Value Theorem

Applies to continuous differentiable functions on interval $a \le x \le b$



There exists a pt. $x=c$ between $x=a$ & $x=b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, the slope of the secant between a & b.

ONLY FOR CONTINUOUS FUNCTIONS (& differentiable)

Use Mean Value Theorem to prove L'Hopital's Rule

L'Hopital was a rich man who hired one of the Bernouli brothers to credit him with one of the inventions. He hired them to tutor him, and then name the theorem after him.

Let $f(x)$ & $g(x)$ be continuous & differentiable.

$$f(a) = g(a) = 0$$
$$g'(a) \neq 0$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

From the MVT, we know that $f(x) = \cancel{f(a)} + f'(c)(x-a)$ for some c between x & a

both = 0

$$f(c) = \frac{f(b)-f(a)}{b-a}$$

$$b = x \Rightarrow f(x) = (x-a)f'(c) + f(a)$$

$$g(x) = \cancel{g(a)} + g'(d)(x-a)$$
for some d between x & a

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)(x-a)}{g'(d)(x-a)} = \lim_{x \to a} \frac{f'(c)}{g'(d)} = \frac{f'(a)}{g'(a)}$$

set them = to a b/c as the graph approaches a, you can set them = to a & plug it in. i love limits.

<u>Now</u>: Assume $f(a) = f'(a) = f''(a) = \ldots f^{(n)}(a) = 0$

$$g(a) = g'(a) = g''(a)$$

$g(a) = g'(a) = g''(a)$
$= \dots g^{(n)}(a) = 0$
& $g^{(n+1)}(a) \neq 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)}$$

Use Taylor's Theorem to Show this.

Taylor's Theorem says: $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}}_{\text{error term}}$

cancel everything & get $f(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$

c is between x & a

$g(x) = \frac{1}{(n+1)!} g^{(n+1)}(d)(x-a)^{n+1}$

d is between x & a

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}}{\frac{1}{(n+1)!} g^{(n+1)}(d)(x-a)}$$

$$= \lim_{x \to a} \frac{f^{(n+1)}(c)}{g^{(n+1)}(d)} = \lim_{x \to a} \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)}$$

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$$ plug it in & get $\frac{0}{0}$

$\frac{d}{dx}(\sin(x) - x) = \cos x - 1$

$\frac{d}{dx}(x^3) = 3x^2$

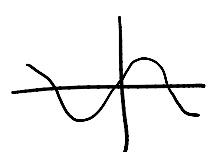
L'Hopital's rule tells us that the ratio, $\frac{\cos x - 1}{3x^2}$ is = to that lim

$$\lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{\sin(x) - x}{x^3}$$

$$= \lim_{x \to 0} \frac{-\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-\cos x}{6}$$ plug it in $\frac{-(\cos 0)}{6} = \boxed{\frac{-1}{6}}$ Answer.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$



odd function. so it only has odd terms in its taylor series.

$$\lim_{x \to 0} \frac{\sin(x) - x}{x^3} = \lim_{x \to 0} \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} \dots}{x^3}$$

$$= -\frac{1}{3!} = \frac{1}{6}$$

sin x always converges, w/ infinite # of terms. given a value

sin x always converges, w/ infinite # of terms. given a value of x. x = 1E6. plug in 1E6 for x, & get sin x in the end. Only if you don't truncate it.

Taylor Series - about $x=a$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

if $a=0$ MacLaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$$

$f(x) = e^x$ $\quad f(0)=1$

$f'(x) = e^x$ $\quad f'(0)=1$

$f''(x) = e^x$ $\quad \vdots$

$f^{(n)}(x) = e^x$ $\quad f^{(n)}(0)=1$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

$\cos(x) = f(x)$ $\quad f(0)=1$ $\quad$ Full Circle

$\sin(x) = f'(x)$ $\quad f'(0)=0$

$-\cos(x) = f''(x)$ $\quad f''(0)=-1$

$\sin(x) = f^{(3)}(x)$ $\quad f^{(3)}(0)=0$

$\cos(x) = f^{(4)}(x)$ $\quad f^{(4)}(0)=1$

and you can check this w/ this sequence

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$f(x) = \sin x$ $\quad f(0)=0$

$f'(x) = \cos x$ $\quad f(0)=1$

$f''(x) = -\sin x$ $\quad f(0)=0$

$f^{(3)}(x) = -\cos x$ $\quad f(0)=-1$

$f^{(4)}(x) = \sin x$ $\quad f(0)=0$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$e^{iy} = \cos y + i \sin y$

$$f(x) = \cancel{f(0)}^{\,0} + \cancel{f'(0)}^{\,1} x + \frac{1}{2}\cancel{f''(0)}^{\,0} x^2 + \dots$$

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots$

$e^{iy} = 1 + iy + \frac{1}{2}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 \dots$

$e^{iy} = 1 + iy - \frac{1}{2}y^2 - \frac{iy^3}{3!} + \frac{1}{4!}y^4 + \dots$

$e^{iy} = \left(1 - \frac{1}{2}y^2 + \frac{1}{4!}y^4 \dots\right) + i\left(y - \frac{y^3}{3!} + \dots\right)$

$$\underbrace{\qquad\qquad}_{\cos y} \qquad \underbrace{\qquad\qquad}_{i\sin y}$$

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$
$$= e^{ia}e^{ib}$$

Reimann, Trapezoidal, Simpsons

Friday, September 29, 2006
9:06 AM

$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\sin(A+B) = \sin A \cos B + \sin B \cos A$

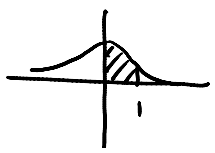
$$e^{i(A+B)} = e^{iA} e^{iB} = \boxed{\cos(A+B) + i\sin(A+B)}$$
$$= (\cos A + i\sin A)(\cos B + i\sin B)$$
$$= \cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin A\sin B$$
$$= (\cos A\cos B - \sin A\sin B) + i(\cos A\sin B + \sin A\cos B)$$
$$= \cos(A+B) + i\sin(A+B)$$

TRUE FOR ALL A & B

$e^{i\pi} = -1$
$e^{i\frac{\pi}{2}} = i$

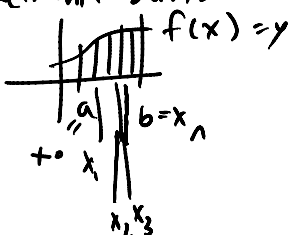
You can get all these strange identities out of this.

$\int_0^1 e^{-x^2}\,dx$



e^{-x^2} is the bell curve.

Reimann Sums



h = width of subinterval $\frac{(b-a)}{n}$

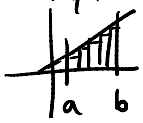
$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} h(f(x_i))$$ left endpt.

$x_i = a + ih$

$h(f(x_0)) + h(f(x_1)) + \ldots + h(f(x_{n-1}))$

Right endpoint $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} h(f(x_i))$

example to estimate the error



line of slope m

there are n triangles & we're missing the area of those.

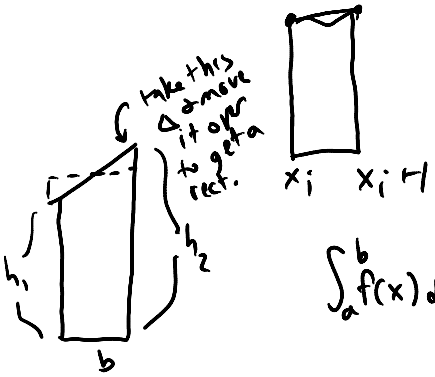
error in subinterval $(x_i + x_{i+1})$
$= \frac{1}{2}$ base · height
$= \frac{1}{2} h\,(f(x_{i+1}) - f(x_i))$
$= \frac{1}{2} h(mh)$ b/c $m = \frac{\text{rise}}{h}$
$= \frac{1}{2} mh^2$ $\quad mh = \text{rise}$

Total error is $(n\,\frac{1}{2}mh^2)$
Total error is $\left(\frac{b-a}{h}\right)\frac{1}{2}mh^2 = \boxed{\frac{b-a}{2}mh}$

The error decreases w/ h linearly. $|E|_x = Ch$
Rectangle Approximation $E \sim h$ Error scales w/ h
Trapezoid Rule $E \sim h^2$ (slower)

Trapezoid Rule

$E \sim h^2$ (claim)



$$A = \frac{1}{2}h\left(f(x_i) + f(x_{i+1})\right)$$

$$\int_a^b f(x)\,dx = \frac{1}{2}h\left(f(x_0)+f(x_1)\right) + \frac{1}{2}h\left(f(x_1)+f(x_2)\right) + \dots + \frac{1}{2}h\left(f(x_{n-1})+f(x_n)\right)$$

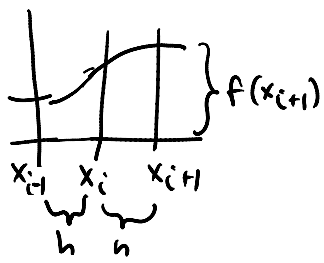
$$= \frac{1}{2}h\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$

$$= \frac{1}{2}h\left[\text{"endpts"} + 2\text{"midpoints"}\right]$$

Simpson's Rule

☆ n must be even

Kind of random — no one knows where it really came from, but it works



Area associated w/ subinterval (x_{i-1}, x_{i+1})

$$\approx \frac{2h}{6}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \left(f(x_2) + 4f(x_3) + f(x_4)\right) + \left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)\Big]$$

$$= \frac{h}{3}\left(\text{"endpts"} + 4(\text{odd interior pts}) + 2(\text{even interior pts})\right)$$

Claim $|E_s| \le C h^4$ Error for Simpson's.

Estimate error for 1 subinterval $x_i = 0$

E_i = error for i^{th} subinterval

use Taylor Approximation

$$E_i = \int_{-h}^{h} f(x)\,dx - \frac{h}{3}\left[f(-h) + 4f(0) + f(h)\right]$$

Use Taylor's Theorem, $a = 0$

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{(4)}(c)x^4$$

c is between 0 & x

$$\int_{-h}^{h}\left[f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)\right]dx$$

$$= 2hf(0) + f''(0)\frac{h^3}{3}$$

$|\text{Error}| \le M$ constant

depends on $f^{(4)}(x)$ between 0 & x.

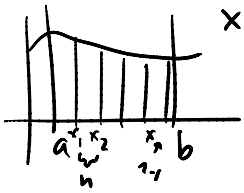
2006.10.2

Simpson's rule

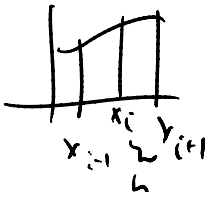
Simpson's rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\dots+f(x_n)\Big]+E_h$$

$x_0=a$
$x_n=b$



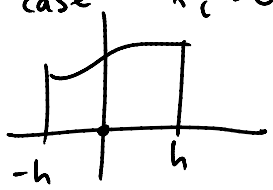
n must be even for this to work.
odd subscript terms have a 4 in front
even subscript terms have a 2 in front



$$A_i=\frac{h}{3}\left[f(x_{i-1})+4f(x_i)+f(x_{i+1})\right]$$

Claim: $|E_h| \le Ch^4$

Special case $x_i=0$



error ends up being h^5, so you have h^5 on each interval & w/ b-a intervals you have h^4, not h^5

Estimate

$$\int_{-h}^{h} f(x)\,dx$$

$$\frac{h}{3}\left[f(-h)+4f(0)+f(+h)\right]$$

Using Taylor's theorem.

$$f(x)=f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+\frac{1}{4!}f^{(4)}(c)x^4$$

c is between 0 & x

You just know that b/c Professor Silber knows that. ACCEPT IT OR ELSE!

$$\int_{-h}^{h} f(x)\,dx=\int_{-h}^{h} f(0)+f'(0)x+\frac{1}{2}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+\frac{1}{4!}f^{(4)}(c)x^4\,dx$$

Assume $f^{(4)}(x) \le$ some M for $x\in(-h,h)$

$$\int_{-h}^{h} f(x)\,dx = f(0)\Big]_{-h}^{h}+f'(0)\frac{x^2}{2}\Big]_{-h}^{h}+\frac{1}{2}f''(0)\frac{x^3}{3}\Big]_{-h}^{h}+\frac{1}{3!}f'''(0)\frac{x^4}{4}+$$

Error, $|\text{Error}| \le \frac{M}{4!}\frac{x^5}{5}\Big]_{-h}^{h}$

$$=2hf(0)+0+\frac{1}{2}f''(0)\frac{1}{3}(2h^3)+0+\text{Error}$$

$|\text{Error}| \le \frac{M}{4!}2h^5$

$$\int_{-h}^{h} f(x)\,dx = 2hf(0)+\frac{1}{3}f'(0)h^3+\text{Error}$$

$|\text{Error}| \le \frac{M}{5!}(2h^5)$

Numerical Approx

$$\frac{h}{3}\left[f(-h)+f(0)+f(h)\right]$$ Replace function by Taylor series

$$= \frac{h}{3}\left[f(0)+\underbrace{f'(0)(-h)}_{\text{cancel}}+\frac{1}{2}f''(0)h^2+\frac{1}{3!}\underbrace{f'''(0)(-h)^3}_{\text{cancel}}+\frac{1}{4!}f^{(4)}(c)h^4\right] \quad c\in(-h,0)$$

$$+\frac{h}{3}\left[4f(0)\right]$$

$$+\frac{h}{3}\left[f(0)+\underbrace{f'(0)h}_{\text{cancel}}+\frac{1}{2}f''(0)h^2+\underbrace{\frac{1}{3!}f'''(0)h^3}_{}+\frac{1}{4!}f^{(4)}(d)h^4\right] \quad d\in(0,h)$$

$$\frac{h}{3}\left[f(-h)+4f(0)+f(h)\right] = 2hf(0)+\frac{1}{3}f''(0)h^3+\text{Error}$$

$$|\text{Error}| \le \frac{M}{3\cdot 4!}h^5$$

b/c you multiply by $\frac{h}{3}$ so $\frac{h^3}{3}$.

Error in one term $\propto h^5$

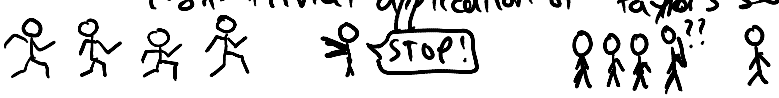
There are n terms & $n=\frac{b-a}{h}$ in series

Total error in simpson's is $\le h^4$

$$\left(\frac{b-a}{h}\right)\left(h^5\right)=(b-a)h^4$$

If you were integrating a cubic there would be no error.

Non-trivial application of taylor's series.



Separable multivariable

Monday, October 02, 2006
9:03 AM

$\frac{dy}{dx} = f(y)g(x)$ <u>separable form.</u>

$M\frac{dv}{dt} = Mg - \alpha v$

$y = v \qquad f(v) = Mg - \alpha v$

$x = t \qquad g(t) = 1$ b/c this is the whole thing

$\frac{dy}{dx} = f(y)g(x)$

$\int \frac{dy}{dx}\frac{1}{f(y)}dx = \int g(x)\,dx$

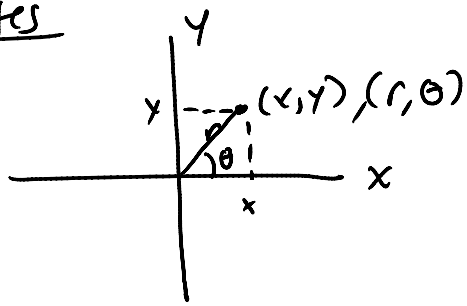
$\int \frac{1}{f(y)}dy = \int g(x)\,dx$ This is how you solve a separable equation.

Differential equation relates a function to its derivatives.

2 Polar Coordinates, Area

Tuesday, October 03, 2006
9:02 AM

Polar coordinates



$x = r\cos\theta$
$y = r\sin\theta$

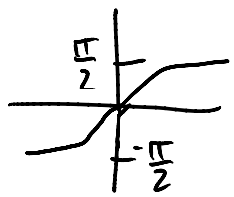
$x^2 + y^2 = r^2$

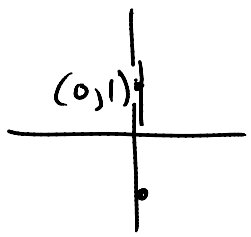
$\frac{y}{x} = \tan(\theta)$

$r = \cancel{\sqrt{x^2+y^2}}$ ≥ 0

$\theta = \cancel{\tan^{-1}\left(\frac{y}{x}\right)}$

b/c we'll never get to ≥ 0 if $x < 0$





$r = 1$	$r = -1$	$r = -1$
$\theta = \frac{\pi}{2}$	$\theta = \frac{3\pi}{2}$	$\theta = -\frac{\pi}{2}$

Graph the following curve $r = 2\cos\theta$

check that it's a circle

$r^2 = 2r\cos\theta$

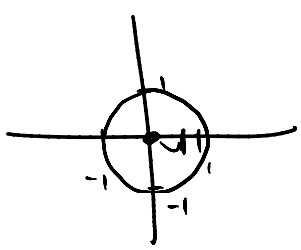
$x^2 + y^2 = 2x$

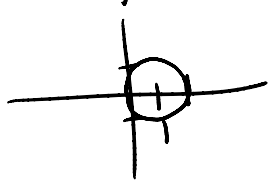
$x^2 - 2x + y^2 = 0$

$(x^2 - 2x + 1) + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Equation for a circle of radius 1 centered at $x=1, y=0$ $(1, 0)$



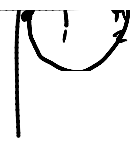


θ	$r = 2\cos\theta$
0	2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2} = \sqrt{3}$
$\pm\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} = \sqrt{2}$
$\pm\frac{\pi}{3}$	$= 1$
$\pm\frac{\pi}{2}$	$= 0$

should be a circle

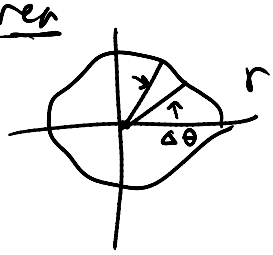
If it's $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, you don't keep going.

$\pm\frac{2\pi}{3}$	$=-1$
$\pm\frac{3\pi}{4}$	$=-\sqrt{2}$



twice

Area



$r=f(\theta)$

approximate area of sector of width $\Delta\theta$ & add them all up.

$\Delta A \approx \frac{1}{2}r^2 d\theta$

imagine you have a $\odot$ of that radius & find area. $\Delta A = (\pi r^2)\left(\frac{\Delta\theta}{2\pi}\right)$ fraction of the full circle you're considering

$\Delta A = \frac{1}{2}r^2 d\theta$

$\Delta A = \frac{1}{2}bh = \frac{1}{2}r\Delta\theta r = \frac{1}{2}r^2 d\theta$

$r\Delta\theta$ r

arc length $= r\Delta\theta$

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

$r=f(\theta)$

Area of circle $r=2\cos\theta$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} 4\cos^2\theta\, d\theta$$

$$A = 2\,\frac{1}{2}\int_0^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

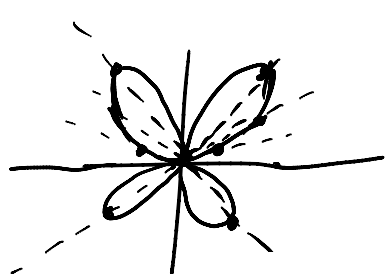
$$2\int_0^{\frac{\pi}{2}} 1+\cos 2\theta\, d\theta$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2\,\frac{\pi}{2} = \pi \checkmark$$

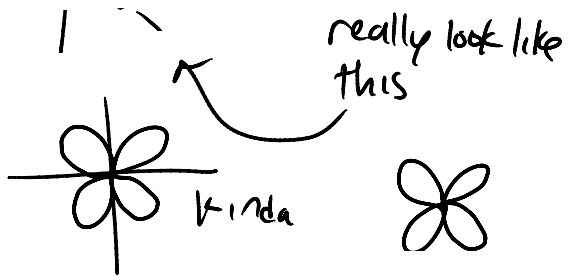
Find the area outside the circle $r=\frac{\sqrt{3}}{2}$ & inside the curve $r=\sin(2\theta)$

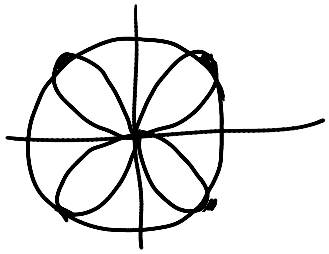
θ	$r=\sin(2\theta)$
0	0
$\pm\frac{\pi}{12}$	$\pm\frac{1}{2}$
$\pm\frac{\pi}{6}$	$\pm\frac{\sqrt{3}}{2}$
[illegible]	[illegible]



4-leaf clover that doesn't really look like

$\pm\frac{\pi}{6}$	$-\frac{\sqrt{3}}{2}$
$\pm\frac{\pi}{4}$	± 1
$\pm\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{4}$	$+1$





points of intersection

$$A = 4\int$$

Pts of intersection : $r = \frac{\sqrt{3}}{2} = \sin 2\theta$

$$2\theta = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{3}$$

$$A = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\left[r_{out}^2 - r_{in}^2\right] d\theta$$

$$A = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\underbrace{\sin^2(2\theta)}_{\frac{1}{2}\left[1-\cos 4\theta\right]} - \frac{3}{4}\right] d\theta$$

$\cos(2(2\theta))$

You need to know double angle formulas & the identity

$$\cos^2\theta + \sin^2\theta = 1$$

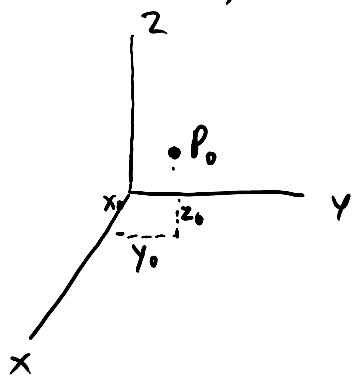
$$\cos^2\theta = \frac{1}{2}\left[1 + \cos 2\theta\right]$$

Vectors

Thursday, October 05, 2006
9:25 AM

Scalars: real #: Magnitude only pressure, temp, distance, speed, time, length, mass...

vectors: Magnitude & direction velocity, acceleration, force, momentum, jerk (derivative of acceleration), torque, angular momentum,



$P_0 = (x_0, y_0, z_0)$

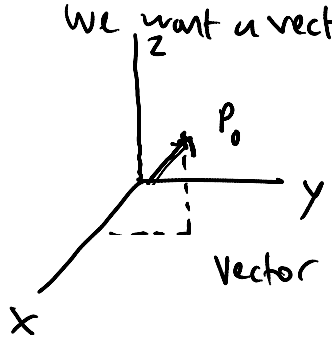
Right-handed coordinate system.

distance between a pt. $P_0 = (x_0, y_0, z_0)$ & $P_1 = (x_1, y_1, z_1)$

$$|P_0P_1| = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2} = \text{scalar.}$$

= scalar.

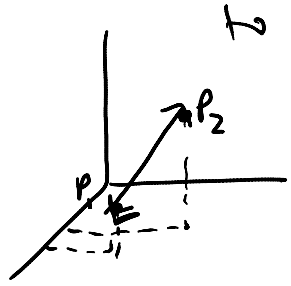
We want a vector from origin to P_0



Origin $O = (0,0,0)$
Point $P_0 = (x_0, y_0, z_0)$

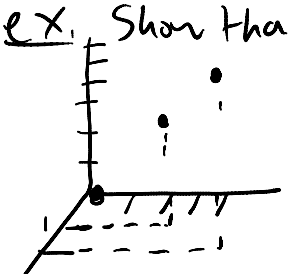
Vector $\overrightarrow{OP} = \langle x_0, y_0, z_0 \rangle$ components of a vector

ex. Vector from $P_1 = \langle x_1, y_1, z_1 \rangle$
to $P_2 = \langle x_2, y_2, z_2 \rangle$

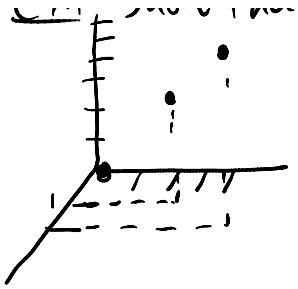


$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

ex. Show that $\overrightarrow{OP_1} = \overrightarrow{P_1P_2}$ when $P_1 = (1,2,3)$ & $P_2 = (2,4,6)$



$\overrightarrow{OP_1} = \langle 1, 2, 3 \rangle$
$\overrightarrow{P_1P_2} = \langle 2-1, 4-2, 6-3 \rangle$
$\overrightarrow{P_1P_2} = \langle 1, 2, 3 \rangle$ so $= \overrightarrow{OP_1}$



$\overrightarrow{OP_1} = \langle 1, 2, 3 \rangle$

$\overrightarrow{P_1P_2} = \langle 2-1, 4-2, 6-3 \rangle$

$\overrightarrow{P_1P_2} = \langle 1, 2, 3 \rangle$ so $= \overrightarrow{OP_1}$

<u>Magnitude</u> of a vector is its length
(length of displacement)

If $\vec{a} = \langle a, b, c \rangle$, then mag. $\vec{a} = |\vec{a}| = \sqrt{a^2+b^2+c^2}$

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

<u>Basic properties of vectors</u>

1) multiplication by a scalar s

$s\vec{a} = s\langle a, b, c \rangle = \langle sa, sb, sc \rangle$

if $s > 0$, then $s\vec{a}$ a point in same direction

if $s < 0$, then $s\vec{a}$ a point in opp. directions.

Doesn't change the direction unless s = neg. Changes magnitude

$|s\vec{a}| = |s|\,|\vec{a}|$

2) Add & Subtract vectors

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{u} + \vec{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$

$\vec{u} - \vec{v} = \langle u_1-v_1, u_2-v_2, u_3-v_3 \rangle$

2006. 10. 6
Friday, October 06, 2006
9:04 AM

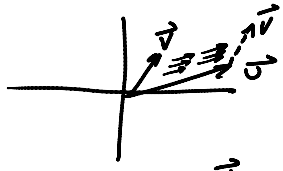
We think of a vector as a displacement from 1 pt in space to another.

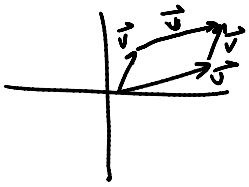
We want to add vectors,

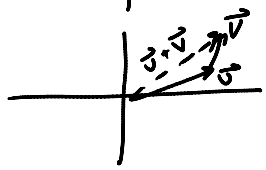
<u>Vector addition & Parallelograms</u> 

2D vectors

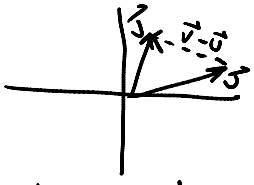
$\vec{u} + \vec{v} = \vec{v} + \vec{u}$
$\vec{u} - \vec{v} = -(\vec{v} - \vec{u})$







$\vec{u}+\vec{v}$ lies on one diagonal on the parallelogram.



$\vec{u} + \vec{v} - \vec{u} = \vec{v}$
$\vec{v} - \vec{u}$ = vector from $\vec{u}$ to $\vec{v}$
$\vec{u} - \vec{v}$ = vector from $\vec{v}$ to $\vec{u}$

<u>unit vectors</u> have length = 1 They indicate direction. Denoted by hats. $|\hat{u}| = 1$

unit vector in x direction $\langle 1,0,0\rangle$ $\hat{i}$ (i-hat) $= (\hat{x})$
unit vector in y direction $\langle 0,1,0\rangle$ $\hat{j}$ (j-hat) $= (\hat{y})$
unit vector in z direction $\langle 0,0,1\rangle$ $\hat{k}$ (k-hat) $= (\hat{z})$

$$\hat{u} = \langle u_1, u_2, u_3\rangle = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$
$$|\hat{u}| = 1$$
$$\sqrt{u_1^2 + u_2^2 + u_3^2} = 1$$

To construct a unit vector, you just divide a vector by its magnitude to get magnitude 1.

If $\vec{A} = \langle A_1, A_2, A_3\rangle \neq \vec{0} = \langle 0,0,0\rangle$
then $\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \left\langle \frac{A_1}{\sqrt{A_1^2+A_2^2+A_3^2}}, \frac{A_2}{\sqrt{A_1^2+A_2^2+A_3^2}}, \frac{A_3}{\sqrt{A_1^2+A_2^2+A_3^2}} \right\rangle$

check that $|\hat{A}| = 1$

$$|\hat{A}| = \sqrt{\frac{A_1^2}{A_1^2+A_2^2+A_3^2} + \frac{A_2^2}{A_1^2+A_2^2+A_3^2} + \frac{A_3^2}{A_1^2+A_2^2+A_3^2}}$$
$$|\hat{A}| = \sqrt{\frac{A_1^2+A_2^2+A_3^2}{A_1^2+A_2^2+A_3^2}}$$
$$|\hat{A}| = 1$$

Differentiation & Integration of vectors

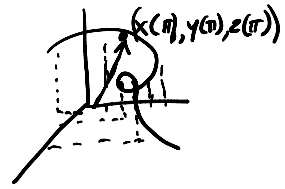
First: Parametric representations of curves

Parameter = t, time.

specify $x = x(t)$ $\quad t_0 \le t \le t_1$

$y = y(t)$

$z = z(t)$

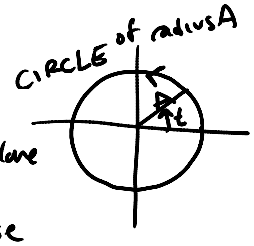


Position vector → you can have a vector $\vec{r}(t)$ pointing to $\langle x(t), y(t), z(t)\rangle$

ex.

$\vec{r} = \langle A\cos(t), A\sin(t), 0\rangle$

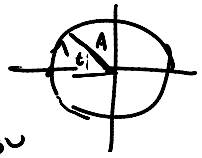
$$\left.\begin{aligned} x &= A\cos(t) \\ y &= A\sin(t) \\ z &= 0 \end{aligned}\right\} \text{curve in } x,y \text{ plane}$$



we start at $t=0$, so counterclockwise

to make it go clockwise, take neg. of 1 of 2 coordinates. Either one.

$\vec{r} = \langle A\cos t, A\sin t, 0\rangle$



if you put a (−) on both, you still go ↖ but you start from π, not 0.

Differentiating vectors

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, t = time

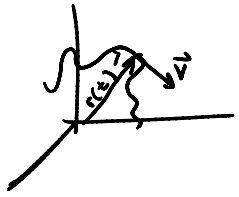
$\frac{d\vec{r}}{dt} = \vec{v}(t)$ b/c time rate of change of position is velocity.

$= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

$= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$\vec{v}$ has magnitude & direction.

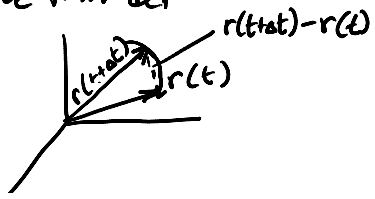
↓ speed $\quad |\vec{v}| = \text{speed}$



$\vec{v}$ points tangent to the velocity curve. So tangent to the curve = direction of $\vec{v}$.

To see that $\vec{v}$ is tangent to the curve from def

$$\lim_{\Delta t \to 0} \underbrace{\frac{r(t+\Delta t) - r(t)}{\Delta t}}_{\frac{\text{displacement in time } \Delta t}{\Delta t}}$$



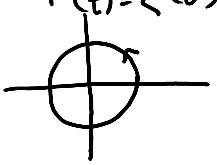
in $\lim_{\Delta t \to 0}$, $r(t+\Delta t) - r(t)$ points tangent to the curve.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right\rangle$$

$\theta = 2\pi = \omega T$; $T = \frac{2\pi}{\omega}$

ex.

$r(t) = \langle \cos(\omega t), \sin(\omega t), 0 \rangle$



$$\vec{v} = \frac{d\vec{r}}{dt} = \langle -A\omega \sin(\omega t), A\omega \cos(\omega t), 0 \rangle, A=1$$

$$|\vec{v}| = \sqrt{A^2\omega^2 \sin^2(\omega t) + A^2\omega^2 \cos^2(\omega t)}$$

$$|\vec{v}| = |\omega| A$$

so speed is ω

that's how quickly you go around the curve.

if $t=0$, no x-component. only have y component in ωA direction.

$\vec{r}(t) \perp \vec{v}(t)$

$$\vec{a} = \frac{d\vec{v}}{dt} = \langle -A\omega^2 \cos(\omega t) - A\omega^2 \sin(\omega t), 0 \rangle$$

$$= -\omega^2 \vec{r}(t)$$

so $\vec{a}$ points in opp direction to $\vec{r}$. Accelerating inwards

Test Review

Monday, October 09, 2006
8:55 AM

1. Find 1st 2 nonzero terms in the Taylor Series expansion about $x=0$ of $y = \operatorname{erf}(x)$, where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$
(you don't need to evaluate the integral)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$
$$\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$
$$\operatorname{erf}''(x) = \frac{2}{\sqrt{\pi}}(-2x)(e^{-x^2}) = \frac{2}{\sqrt{\pi}}(-2x)(e^{-x^2})$$
$$\operatorname{erf}'''(x) = \frac{2}{\sqrt{\pi}}\left[-2(e^{-x^2}) - 2x(-2x)(e^{-x^2})\right]$$

Taylor series expansion: $f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 \ldots$

$a=0$

$\operatorname{erf}(0) = 0$

$\operatorname{erf}'(0) = \frac{2}{\sqrt{\pi}}$

$\operatorname{erf}''(0) = 0$

$\operatorname{erf}'''(0) = -\frac{4}{\sqrt{\pi}}$

$$T = 0 + \frac{2}{\sqrt{\pi}}(x) + \frac{1}{3!}\left(-\frac{4}{\sqrt{\pi}}\right)$$
$$T = \frac{2}{\sqrt{\pi}}x + \frac{x^2}{\sqrt{\pi}}$$

2. Consider curve defined by $y^2 = (x-1)(x-2)$
a) Find all pts on curve where tangent line is horizontal
b) Find all pts where tangent is vertical
c) Show that $(x,y) = (3,\sqrt{2})$ is on the curve & write equation for tangent line to the curve at that pt.

a) $\frac{dy}{dx} = 0$

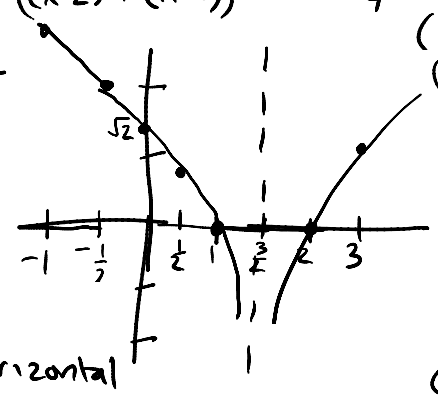
$$y = \sqrt{(x-1)(x-2)}$$
$$\frac{dy}{dx} = \frac{1}{2}\left((x-1)(x-2)\right)^{-\frac{1}{2}}\left((x-2)+(x-1)\right)$$
$$\frac{dy}{dx} = \frac{2x-3}{2\sqrt{(x-1)(x-2)}}$$
$$0 = \frac{2x-3}{2\sqrt{(x-1)(x-2)}}$$
$$0 = 2x-3$$
$$3 = 2x$$
$$\frac{3}{2} = x \quad \text{DNE}$$

Tangent line is not horizontal

$\frac{dx}{dy} = 0$ w/ vertical tangent

or diff implicitly,

$$2y\frac{dy}{dx} = (x-2) + (x-1) = 2x-3$$



$\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)$
$= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$
$= -\frac{1}{4}$

$\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)$
$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$
$\sqrt{\left(\frac{3}{4}\right)} = \frac{\sqrt{3}}{2}$

$\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)$
$\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$
$\sqrt{\frac{15}{4}}$

$(-1-1)(-1-2)$
$(-2)(-3)$
$\sqrt{6}$

$(3-1)(3-2)$
$2(1)$
2

$-7\,dx$ 2

$0 = 2x-3$

$\frac{3}{2} = x$ but it still DNE

b) $\frac{3}{2}$ is vertical asymptote

$\frac{dx}{dy} = 0$

$$2y = (x-2)\frac{dx}{dy} + (x-1)\frac{dx}{dy}$$

$$\frac{2y}{(x-2)+(x-1)} = \frac{dx}{dy}$$

$$\frac{2y}{2x-3} = \frac{dx}{dy}$$

$$\frac{2y}{2x-3} = 0$$

$$y = 0$$

$0 = (x-1)(x-2)$

$0 = x-1$ $\quad$ $x-2=0$

$1 = x$ $\quad$ $x=2$

$(1,0)$ $\quad$ $(2,0)$

c) $(3, \sqrt{2})$

$(\sqrt{2})^2 = (3-1)(3-2)$

$2 = (2)(1)$

$2 = 2$ ✓

$$\frac{dy}{dx} = \frac{2x-3}{2y}$$

$$\left.\frac{dy}{dx}\right|_{\substack{y=\sqrt{2}\\x=3}} = \frac{2(3)-3}{2(\sqrt{2})}$$

$$= \frac{3}{2\sqrt{2}}$$

$$y - \sqrt{2} = \frac{3}{2\sqrt{2}}(x-3)$$

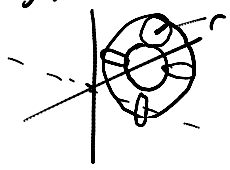
$$y - \sqrt{2} = \frac{3}{2\sqrt{2}}x - \frac{9}{2\sqrt{2}}$$

$$y = \frac{3}{2\sqrt{2}}x - \frac{9}{2\sqrt{2}} + \sqrt{2}$$

3. revolve circle $x^2+(y-b)^2 = r^2$ $(b>r>0)$ about the x-axis, creating a donut (torus).

a) Use method of disks + washers to write an integral for the volume of the torus.

b) Evaluate integral to determine volume.



centered at $(0,b)$ $\quad$ $r=r$

$$V = \int_{b}^{r} \frac{4}{3}\pi\left(\sqrt{x^2+(y-b)^2}\right)^3 dr - \int_{-r}^{b} \frac{4}{3}\pi (b)^3 dr$$

$\frac{4}{3}\pi r^3$

$$V = \pi \int_{-r}^{r} 4b\sqrt{r^2-x^2}\,dx$$

$$= \pi \int (r_{out}^2 - r_{in}^2)\,dx$$

$$= \pi \int_{-r}^{r} \left(r_{out}^2 - r_{in}^2 \right) dx$$

$y = \pm\sqrt{r^2 - x^2} + b$ $= r_{out}$ $(+)$

$= r_{in}$ $(-)$

answer $= 2\pi^2 b r^2$

$$= \pi \int \left(\sqrt{r^2 - x^2} + b \right) - \left(-\sqrt{r^2 - x^2} + b \right) dx$$

Test Review 2

Monday, October 09, 2006
8:55 AM

2. Consider curve defined by $y^2=(x-1)(x-2)$

a) Find all pts on curve where tangent line is horizontal

b) Find all pts where tangent is vertical

c) Show that $(x, y) = (3, \sqrt{2})$ is on the curve & write equation for tangent line to the curve at that pt.

Integration of Vectors

Wednesday, October 11, 2006
8:52 AM

Integration of vectors
done component by component

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$
$$\int \vec{r}(t) = \langle \int x(t), \int y(t), \int z(t)\rangle$$

ex

$$\frac{d\vec{r}}{dt} = \vec{v}(t) = \langle A, B, C\rangle = \text{constant velocity}$$

$$\int_0^T \frac{d\vec{r}}{dt}\,dt = \left\langle \int_0^T A\,dt, \int_0^T B\,dt, \int_0^T C\,dt \right\rangle$$

$$\vec{r}(t)\Big]_0^T = \langle At\Big]_0^T, Bt\Big]_0^T, Ct\Big]_0^T\rangle$$

$$\vec{r}(T) - \vec{r}(0) = \langle AT, BT, CT\rangle$$

$$r(0) = \langle x_0, y_0, z_0\rangle$$

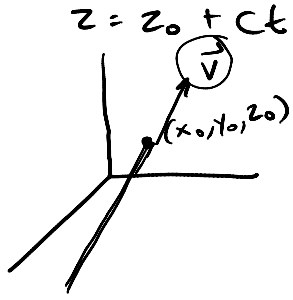
$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle + \langle x_0 + At, y_0 + Bt, z_0 + Ct\rangle$$

You can have a line in 3 dimensions

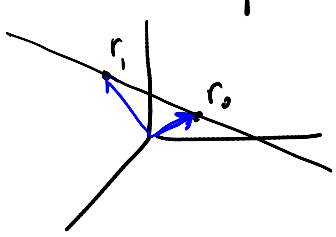
Parametric equation

$$x = x_0 + At$$
$$y = y_0 + Bt$$
$$z = z_0 + Ct$$

goes through point (x_0, y_0, z_0)
and points in direction of $\vec{v} = \langle A, B, C\rangle$



What is the parametric equation for the line through $r_0 = (x_0, y_0, z_0)$
$r_1 = (x_1, y_1, z_1)$



$$\vec{v} = \vec{r}_1 - \vec{r}_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$$

$$\left.\begin{aligned} x &= x_0 + (x_1 - x_0)t \\ y &= y_0 + (y_1 - y_0)t \\ z &= z_0 + (z_1 - z_0)t \end{aligned}\right\} \text{equation for a line.}$$

if $t = 0$, $(x, y, z) = (x_0, y_0, z_0)$
if $t = 1$, $(x, y, z) = (x_1, y_1, z_1)$

if $t=1$, $(x,y,z) = (x_1, y_1, z_1)$

equation for a line (not parametric)

$x = x_0 + At$ $\quad t = \frac{x-x_0}{A}$ $\quad$ assuming $A \neq 0$ if $A=0$, then $x = x_0$ & it's $\parallel$ to the yz plane.

$y = y_0 + Bt$ $\quad t = \frac{y-y_0}{B}$ $\quad$ if $B \neq 0$

$z = z_0 + Ct$ $\quad z = \frac{z-z_0}{C}$ $\quad$ if $C \neq 0$

$$\frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C}$$

$$\boxed{\begin{array}{l} \frac{x-x_0}{y-y_0} = \frac{A}{B} \\ \& \\ \frac{x-x_0}{z-z_0} = \frac{A}{C} \end{array}}$$

plane

plane

each of these equations would define a line.

$A(y-y_0) = (x-x_0)B$

$y = \frac{B(x-x_0)}{A} + y_0$ $\quad$ Now you have line, & this plane has all values of this for y. Then solve for x & z.

intersect then both & you get a line.

<u>Multiplying vectors</u>

Dot product $\quad \vec{A} \cdot \vec{B}$ = scalar

Cross product $\quad \vec{A} \times \vec{B}$ = vector

dot product defined

let $\vec{u} = \langle u_1, u_2, u_3 \rangle$

let $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ $\quad$ <u>Scalar</u>

Properties (easy to prove):

(1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(2) $\vec{a} \cdot (\vec{u} + \vec{v}) = \vec{a} \cdot \vec{u} + \vec{a} \cdot \vec{v}$

(3) $(s\vec{a}) \cdot \vec{b} = \vec{a} \cdot (s\vec{b})$

(4) $\vec{a} \cdot \vec{0} = 0$

(5) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ $\quad$ This is important to proving other things.

<u>Problem</u> $\quad$ Assume $\vec{a} \neq \vec{0}$ & $\vec{b} \neq \vec{0}$

Show that $\vec{a} \cdot \vec{b} = 0$ iff $\vec{a} \perp \vec{b}$ $\quad$ Statement of 2 diff theorems, when iff.

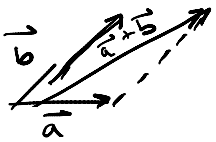
if $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$ $\quad$ $\vec{a}, \vec{b} \neq 0$

if $\vec{a} \perp \vec{b}$, then $\vec{a} \cdot \vec{b} = 0$

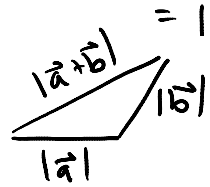
(1) Assume $\vec{a} \cdot \vec{b} = 0$ & show $\vec{a} \perp \vec{b}$

(1) Assume $\vec{a}\cdot\vec{b}=0$ & show $\vec{a}\perp\vec{b}$

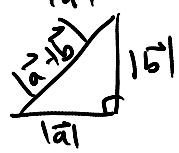
$\vec{a}+\vec{b}$



$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$
$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$
$$= |\vec{a}|^2 + 0 + 0 + |\vec{b}|^2$$
$$= |\vec{a}|^2 + |\vec{b}|^2$$

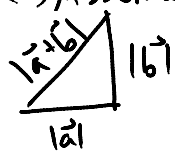


Now this is the Pythagorean Theorem



$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$ only if $\vec{a}\perp\vec{b}$

(2) Assume $\vec{a}\perp\vec{b}$ & show that $\vec{a}\cdot\vec{b}=0$



$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

etc.

Dot Products, Determinants

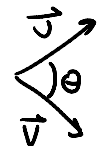
Thursday, October 12, 2006
9:03 AM

Dot Products

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$
& $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ scalar

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$ $\qquad \theta = \measuredangle$ between $\vec{u}$ & $\vec{v}$



Prove that $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\vec{v} - \vec{w}$ b/c $\vec{w}$ + this vector $= \vec{v}$ (vector addition)

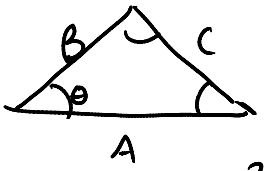
$\vec{w} = \hat{u}$ points in direction of u

$\vec{w} = |\vec{v}|\cos\theta \; \hat{u}$

$\hat{u} = \dfrac{\vec{u}}{|\vec{u}|}$

plug in $|\vec{w}|$

$$|\vec{v}|^2 = |\vec{w}|^2 + |(\vec{v} - \vec{w})|^2$$
$$= |\vec{w}|^2 + (\vec{v} - \vec{w})(\vec{v} - \vec{w})$$
$$= |\vec{w}|^2 + |\vec{v}|^2 - 2\vec{v}\cdot\vec{w} + |\vec{w}|^2$$
$$2\vec{v}\cdot\vec{w} = 2|\vec{w}|^2$$
$$\vec{v}\cdot\vec{w} = |\vec{w}|^2$$
$$\vec{v}\cdot|\vec{v}|\cos\theta \frac{\vec{u}}{|\vec{u}|} = v^2\cos^2\theta \cancel{|\vec{u}|^2}^{1}$$
$$\cancel{\vec{v}^2 \cos^2\theta \, |\vec{u}|^2 = |\vec{w}|^2}$$

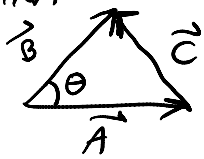


Law of Cosines

$$(\vec{u}\cdot\vec{u})\left(\frac{|\vec{v}|}{|\vec{u}|}\cos\theta\right) = |\vec{u}|^2\cos\theta$$
$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$C^2 = A^2 + B^2 - 2AB\cos\theta$

Derivation:



$\vec{C} = \vec{B} - \vec{A}$

$$|\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{B} - \vec{A})\cdot(\vec{B} - \vec{A})$$
$$\vec{B}\cdot\vec{B} - 2\vec{B}\cdot\vec{A} + \vec{A}\cdot\vec{A}$$
$$= \vec{B}^2 + \vec{A}^2 - 2AB\cos\theta$$

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

Determinants

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad |A| = \mathrm{Det}(A) =$ determinant of matrix A

Is a #. Scalar.

$\mathrm{Det}(A) = ad - bc$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ad $-$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ bc

3x3 matrix determinant

alternate signs

3x3 matrix determinant

alternate signs

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = a\left|\begin{bmatrix} e & f \\ h & j \end{bmatrix}\right| - b\left|\begin{bmatrix} d & f \\ g & j \end{bmatrix}\right| + c\left|\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right|$$

$$= a(ej - fh) - b(dj - fg) + c(dh - eg)$$

4x4 matrix determinant

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

$$m_{11}[\qquad] - m_{12}[\qquad] + m_{13}[\qquad] - m_{14}[\quad]$$

$$m_{11}\left|\begin{bmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{bmatrix}\right| - m_{12}\ e|$$

Cross Products

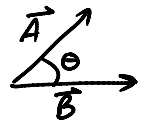
Friday, October 13, 2006
9:03 AM

Geometric def. of a cross product:

$\vec{A} \times \vec{B}$ = vector

$= |\vec{A} \times \vec{B}|\, \hat{n}$

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$



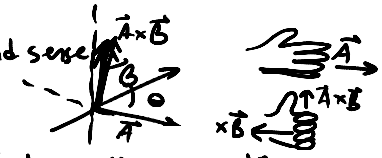
dot product was $|\vec{A}||\vec{B}|\cos\theta$

if 2 vectors are ||, ther cross product is 0.

$\vec{A} \times \vec{B} \perp \vec{A}$

$\vec{A} \times \vec{B} \perp \vec{B}$

in right-hand sense



(1) $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$ ∡ between them is < 180, so you don't need to put $|\theta|$.

(2) Cross product isn't commutative.

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ b/c right hand rule. $\vec{B} \times \vec{A}$ is opp. direct.

(3) $(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot B = 0$ dot product of 2 vectors that are $\perp$ to each other is 0.

(4) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

(5) $\vec{A} \times (\lambda \vec{B}) = (\lambda \vec{A}) \times \vec{B} = \lambda(\vec{A} \times \vec{B})$. λ only changes magnitude.

(6) $\vec{A} \times \vec{A} = \vec{0}$. Angle between a vector + itself = 0, so $\sin 0 = 0$.

(7) $\hat{i} \times \hat{j} = \hat{k}$ x direction, y direction = z direction

(8) $\hat{j} \times \hat{k} = \hat{i}$ y direction, z direction = x direction

(9) $\hat{k} \times \hat{i} = \hat{j}$ z direction, x direction = y direction

(10) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

$\vec{a} = \langle x_1, y_1, z_1 \rangle = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{b} = \langle x_2, y_2, z_2 \rangle = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\vec{a} \times \vec{b} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \times (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$

$\vec{a} \times \vec{b} = (0 + x_1y_2\hat{k} - x_1z_2\hat{j}) + (-y_1x_2\hat{k} + y_1z_2\hat{i})$
$+ (z_1x_2\hat{j} - z_1y_2\hat{i})$ The minuses come in b/c of order:

$\hat{j} \times \hat{i} = -\hat{k}$. $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \times \hat{k} = -\hat{j}$

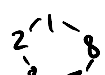
Switch the minus so it's like a determinant

$\vec{a} \times \vec{b} = (y_1z_2 - y_2z_1)\hat{i} - (x_1z_2 - x_2z_1)\hat{j} + (x_1y_2 - y_1x_2)\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \hat{i}(y_1z_2 - z_1y_2) - \hat{j}(x_1z_2 - z_1x_2) + \hat{k}(x_1y_2 - x_2y_1)$$

Determinant of a matrix = cross product by components.

18782
81827 Permutation



Then you can start @ diff points

18782
87821
78218
82187

$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$

Cyclic Permutation

$\hat{i} \times \hat{k} = -\hat{j}$
$\hat{k} \times \hat{j} = -\hat{i}$

Show that the area of the Δ w/ sides $\vec{A}$ & $\vec{B}$ is $\frac{1}{2}|\vec{A} \times \vec{B}|$

$\frac{1}{2}|\vec{A}||\vec{B}|\sin\theta = \text{Area}$

base $= |\vec{A}|$
height $= |\vec{B}|\sin\theta$

$\frac{1}{2}|\vec{A}||\vec{B}|\sin\theta = \frac{1}{2}bh$
$\frac{1}{2}|\vec{A}||\vec{B}|\sin\theta = \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$

Let $\vec{a} = \langle 1,2,3 \rangle$
$\vec{b} = \langle 1,1,1 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(2-3) - \hat{j}(1-3) + \hat{k}(1-2)$$
$$= -\hat{i} + 2\hat{j} - \hat{k}$$
$$= \langle -1, 2, -1 \rangle$$

Check that cross product $\vec{a} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{a} \times \vec{b}) = 0$
$\langle -1,2,-1 \rangle \cdot \langle 1,2,3 \rangle = 0$ ✓
$\langle -1,2,-1 \rangle \cdot \langle 1,1,1 \rangle = 0$ ✓
we've constructed a vector $\perp$ to $\vec{a}$ & $\vec{b}$.

Calculate $\vec{a} \cdot (\vec{b} \times \vec{c})$

$\vec{a} = \langle x_0, y_0, z_0 \rangle$
$\vec{b} = \langle x_1, y_1, z_1 \rangle$
$\vec{c} = \langle x_2, y_2, z_2 \rangle$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$
$$= \hat{i}(y_1 z_2 - z_1 y_2) - \hat{j}(x_1 z_2 - z_1 x_2) + \hat{k}(x_1 y_2 - y_1 x_2)$$

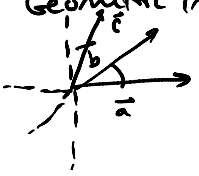
$\vec{a} \cdot (\vec{b} \times \vec{c})$
$= x_0(y_1 z_2 - z_1 y_2) - y_0(x_1 z_2 - z_1 x_2) + z_0(x_1 y_2 - y_1 x_2)$
we just replaced $\hat{i}$, $\hat{j}$, & $\hat{k}$ w/ x_0, y_0, & z_0

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{vmatrix} = \vec{c} \cdot (\vec{a} \times \vec{b})$$

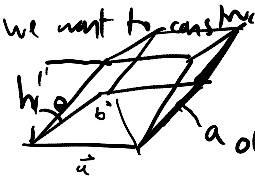
cyclic permutation

So you get that property.

Geometric interpretation



we want to construct a parallelogram 3D figure.
parallelopiped



a of bottom llogram = area

Volume $= \vec{A} \times (\vec{B} \cdot \vec{C})$
$= ah$
$= a|\vec{c}|\cos\theta$
Volume $= |\vec{A} \times \vec{B}||\vec{c}|\cos\theta$

Area of the Δ is $\frac{1}{2}|\vec{A} \times \vec{B}|$
$a = |\vec{A} \times \vec{B}|$

$$\underbrace{\quad}_{\text{mag}} \cdot \underbrace{\quad}_{\text{mag}} \cdot \cos\theta = \text{dot product}$$

$$\text{Volume} = \left| (\vec{A} \times \vec{B}) \cdot \vec{C} \right|$$

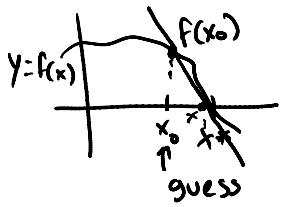
Newton Rapheson

Monday, October 16, 2006
8:55 AM

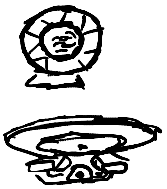
$\boxed{\text{Newton Rapheson}}$

$f(x^*) = 0$

Solve for x



X-FILES



$x^* \approx x_0$

$f(x_0) \neq 0$

$\frac{\text{rise}}{\text{run}} = \frac{f(x_0)}{x_0 - x_1}$

$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$

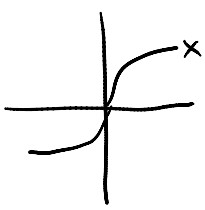
b/c you chose the tangent line

$f(x_0) = f'(x_0)(x_0 - x_1)$

$x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$

x	$f(x)$
x_0	$f(x_0) \approx .01$ or something so plug it in.

$x_1 = x_0 - \frac{.01}{f'(x_0)} \approx$ closer to 0



$x^{\frac{1}{3}}$ infinite slope, horrible function to iterate.

Planes: $\perp$ to particular line

: specify 3 non collinear pts.

Equations that describe planes

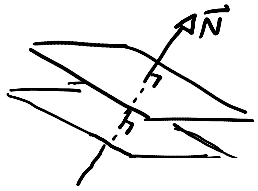
$ax + by + cz = d$

3 equations, 4 unknowns even if you have 3 points.

Equation for a plane

Let $\vec{N} = \langle a, b, c \rangle$ be a vector that is $\perp$ to the plane

$\vec{N}$ = normal vector



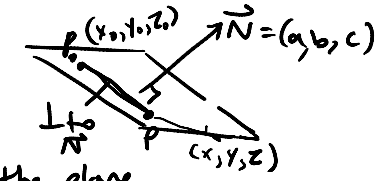
Parallel planes have the same normal vector

To uniquely specify a plane, you also need to know 1 pt. on it.

$P_0 = (x_0, y_0, z_0)$ is in plane

$P(x_0, y_0, z_0)$ $\vec{N} = (a, b, c)$

$P_0 = (x_0, y_0, z_0)$ is in plane



Suppose a pt. (x, y, z) is in the plane

Construct $\overrightarrow{P_0P} = \langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle$

$= \langle x - x_0, y - y_0, z - z_0\rangle$ will be $\perp$ to $\vec{N}$

$\overrightarrow{P_0P} \cdot \vec{N} = |\overrightarrow{P_0P}||\vec{N}| \cos\theta \leftarrow$ if $\perp$, the $\cos\theta = 0$
$\uparrow$
$90°$

$\overrightarrow{P_0P} \cdot \vec{N} = 0$

$\overrightarrow{P_0P} \cdot \vec{N} = \langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle$

$0 = ax - ax_0 + by - by_0 + cz - cz_0$

$\underbrace{ax + by + cz}_{\text{was original equation, } = d} = ax_0 + by_0 + cz_0 = d$

$ax_0 + by_0 + cz_0 = d$, where $\vec{N} = \langle a, b, c\rangle$

it follows that if $P_0 = (x_0, y_0, z_0)$ is in the plane then

$d = ax_0 + by_0 + cz_0$

They choose d = something after solving. d could be anything (in the plane)

<u>Problem</u>: A plane can also be uniquely specified by 3 non-collinear pts in it.

P_0, P_1, + P_2 are all in the plane.

$P_0 = (x_0, y_0, z_0)$

$P_1 = (x_1, y_1, z_1)$

$P_2 = (x_2, y_2, z_2)$

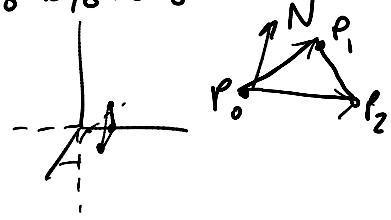
Find $\vec{N}$ + hence obtain equation.

$ax + by + cz = d = ax_0 + by_0 + cz_0$

Let $P_0 = (1, 2, 1)$

$P_1 = (2, 2, 2)$

$P_2 = (3, 2, 1)$



$\overrightarrow{P_0P_1} = \langle 2,2,2\rangle - \langle 1,2,1\rangle$

$\overrightarrow{P_0P_1} = \langle 1,0,1\rangle$

$\overrightarrow{P_0P_2} = \langle 3,2,1\rangle - \langle 1,2,1\rangle$

$\overrightarrow{P_0P_2} = \langle 2,0,0\rangle$

get a vector $\perp$ to plane

for a vector $\perp$ to plane
by taking cross product.

$\vec{N} = \overrightarrow{P_0P_2} \times \overrightarrow{P_0P_1}$ going from $\overrightarrow{P_2P_0}$ across $\overrightarrow{P_0P_1}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (0-0) - (2-0) + (0-0)$$
$$= \langle 0, -2, 0 \rangle$$

$$\cancel{a}^{0}x + \cancel{b}^{-2}y + \cancel{c}^{0}z = d$$
$$-2y = d$$ so pick a y, since all y's satisfy this.
$$-2(2) = d$$
$$-4 = d$$

equation for line in this case is $y = 2$.

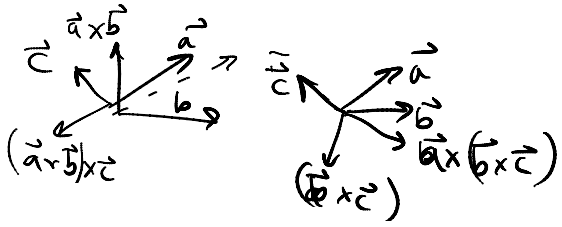
The Dot product will be 0 if 2 vectors are $\perp$. So it's a great check to do if you think your vectors are $\perp$

The cross product will always give you a vector $\perp$ to your original 2 vectors.

Double Cross Product

$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ bac cab rule.

dx I need the parenthases on the left-hand side? is $\vec{a} \times \vec{b} \times \vec{c}$ ambiguous, or = to $(\vec{a} \times \vec{b}) \times \vec{c}$ & $a \times (\vec{b} \times \vec{c})$



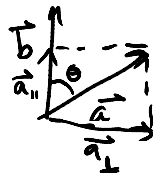
So no, it doesn't work.
$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$
bac cab isn't consistent with that. And it doesn't work out when you try it.

Vectors

Tuesday, October 17, 2006
8:54 AM

Consider vector $\vec{a}$ & vector $\vec{b} \neq \vec{0}$
we can uniquely express $\vec{a}$ in terms of a vector $\parallel$ to $\vec{b}$
& one $\perp$ to $\vec{b}$

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$$



$$|\vec{a}_{\parallel}| = |\vec{a}| \cos\theta \, \hat{b}$$

$\hat{b}$ tells you direction.

$$\frac{\vec{b}}{|\vec{b}|} = \hat{b}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$|\vec{a}_{\parallel}| = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b} = (\vec{a}\cdot\hat{b})\hat{b} = |\vec{a}_{\parallel}|$$

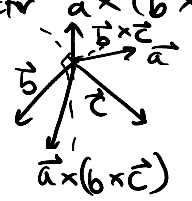
$$\vec{a}_{\perp} = \vec{a} - |\vec{a}_{\parallel}|$$

$$\vec{a}_{\perp} = \vec{a} - (\vec{a}\cdot\hat{b})\hat{b}$$

Proving the double cross product formula:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

The vector $\vec{a} \times (\vec{b} \times \vec{c})$ is in the same plane as the vectors $\vec{b}$ & $\vec{c}$



Mindless Proof by components

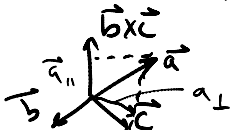
Thinking Proof by geometry

bac cab

Consider a vector $\vec{v} = b(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

Show that (a) $\vec{v}$ pts in direction as $\vec{a} \times (\vec{b} \times \vec{c})$
(b) $\vec{v}$ has the same mag as

Let $\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$ where $\vec{a}_{\parallel}$ is in the direction of $\vec{b} \times \vec{c}$
& $\vec{a}_{\perp}$ is $\perp$ to $\vec{b} \times \vec{c}$



$\vec{a}_{\perp}$ is in the $\vec{b}$ $\vec{c}$ plane

Claim $\vec{v} = \vec{b}(\vec{a}_{\perp}\cdot\vec{c}) - \vec{c}(\vec{a}_{\perp}\cdot\vec{b})$

$\vec{a}_{\parallel}$ is $\perp$ to $\vec{b}$ & $\vec{c}$
$\vec{a}_{\parallel}\cdot\vec{c} = 0$
$\vec{a}_{\parallel}\cdot\vec{b} = 0$

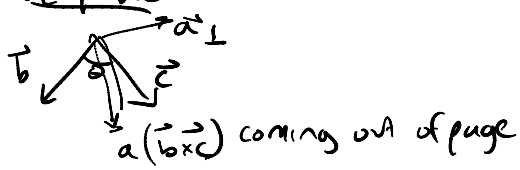
$$\vec{v} = \vec{b}(\vec{a}_{\perp}\cdot\vec{c}) - \vec{c}(\vec{a}_{\perp}\cdot\vec{b})$$

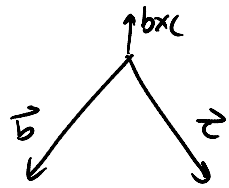
So you can replace

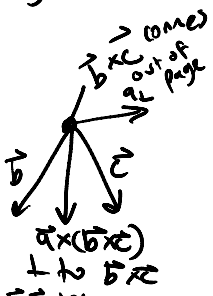
$\vec{v} = \vec{b}(\vec{a}_\perp \cdot \vec{c}) - \vec{c}(\vec{a}_\perp \cdot \vec{b})$ So you can replace all $\vec{a}$'s w/ just $\vec{a}_\perp$.

(a) We know that $\vec{a}\times(\vec{b}\times\vec{c})$ is in the $\vec{b}\vec{c}$ plane.
And $\vec{v}$ is in that plane
Show $\vec{a}\times(\vec{b}\times\vec{c})$ is $\perp$ to $\vec{a}$.

$\vec{b}\vec{c}$ plane







$a_\perp$ in $\vec{b}\vec{c}$ plane b/c a comes out of page at an $\angle$.
b/c $\vec{b}$ & $\vec{c}$ are on same plane & $\vec{a}_\perp$ is $\perp$ to $\vec{b}\vec{c}$ plane.

Show that $\vec{v} \perp \vec{a}$

$$\vec{v}\cdot\vec{a}_\perp = \vec{b}(\vec{a}_\perp\cdot\vec{c})\cdot\vec{a}_\perp - \vec{c}(\vec{a}_\perp\cdot\vec{b})\cdot\vec{a}_\perp)$$
$$= (\vec{a}_\perp\cdot\vec{c})(\vec{b}\cdot\vec{a}_\perp) - (\vec{a}_\perp\cdot\vec{b})(\vec{c}\cdot\vec{a}_\perp) = 0$$

$\vec{v}$ is in the same plane ($\vec{b}\vec{c}$) & $\perp$ to $\vec{a}$.

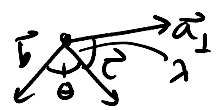
And therefore it is either $\parallel$ or anti $\parallel$ to $\vec{a}\times(\vec{b}\times\vec{c})$

Examine $\vec{v}\times\vec{a}_\perp$

$$|\vec{v}\times\vec{a}_\perp| = |\vec{v}||\vec{a}_\perp|\sin\lambda$$

if $\vec{v}$ is $\parallel$ to $\vec{a}\times(\vec{b}\times\vec{c})$ then in what direction is $\vec{v}\times\vec{a}_\perp$?

Same direction as $(\vec{b}\times\vec{c})$



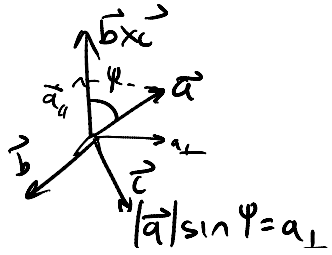
$$\vec{v}\times\vec{a}_\perp = (\vec{a}_\perp\cdot\vec{c})\vec{b}\times\vec{a}_\perp - (\vec{a}_\perp\cdot\vec{b})\vec{c}\times\vec{a}_\perp$$
$$= \hat{n}\left[(|\vec{a}_\perp||\vec{c}|\cos\lambda)|\vec{b}||\vec{a}_\perp|\sin(\lambda+\theta)\right] - \left[|\vec{a}_\perp||\vec{b}|\cos(\lambda+\theta)(|\vec{c}||\vec{a}_\perp|\sin\lambda)\right]$$

unit vector pointing in direction of $\vec{b}\vec{c}$ plane

$$= \hat{n}\left[|\vec{a}_\perp|^2|\vec{b}||\vec{c}|(\underbrace{\cos(\lambda)\sin(\lambda+\theta)}_{\cos A\,\sin B} - \underbrace{\cos(\lambda+\theta)\sin\lambda}_{-\cos B\,\sin A})\right]$$
$$= \hat{n}\left[|\vec{a}_\perp|^2|\vec{b}||\vec{c}|\underbrace{\sin(\lambda+\theta-\lambda)}_{\sin(B-A)}\right]$$
$$= \hat{n}\left[|\vec{a}_\perp|^2|\vec{b}||\vec{c}|\sin\theta\right]$$

$\hat{n}$ means pointing in right direction.

$$|\vec{v}| = |\vec{a}\times(\vec{b}\times\vec{c})|$$
$$|\vec{v}\times\vec{a}_\perp| = |\vec{v}||\vec{a}_\perp| = |\vec{a}_\perp|^2|\vec{b}||\vec{c}|\sin\theta$$
$$|\vec{v}| = |\vec{a}_\perp||\vec{b}||\vec{c}|\sin\theta$$



$$\vec{a}\times(\vec{b}\times\vec{c}) = |\vec{a}||\vec{b}\times\vec{c}|\sin\varphi$$
$$\vec{a}\times(\vec{b}\times\vec{c}) = |\vec{a}_\perp||\vec{b}\times\vec{c}|$$
$$= |\vec{a}_\perp||\vec{b}||\vec{c}|\sin\theta = |\vec{v}|$$

so $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{v} = |\vec{a}_\perp||\vec{b}||\vec{c}|\sin\theta$

Sketching Curves

Wednesday, October 18, 2006
9:15 AM

Sketching Curves

Suppose we have a function $f(x)$

To Sketch $f(x)$,

1. Find the critical points
   i.e. Solve $f'(x)=0$, find where $f'(x)$ DNE
2. Determine where $f(x)$ ↑ & ↓.
3. Find inflection points
   i.e. Solve $f''(x)=0$, find where $f''(x)$ DNE
4. Determine where $f(x)$ is concave up / down — $f''(x)>0$ concave up, $f''(x)<0$ concave down
5. Find x & y intercepts
6.
7. Find any asymptotes, discontinuities, and behavior of f as $x \to \pm\infty$
   And behavior of f near any discontinuities.
8. Join the points, determine the curve.

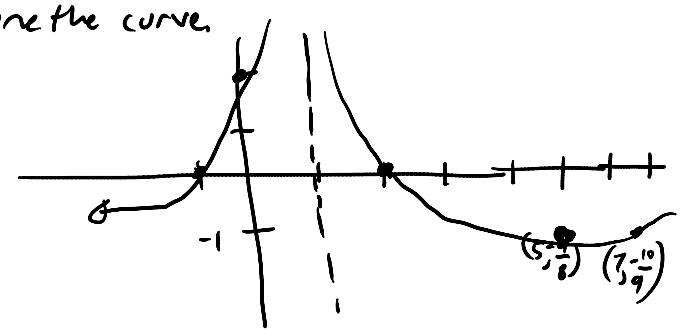
$$f(x) = \frac{2+x-x^2}{(x-1)^2}$$

Discontinuity at $x=1$

$$\lim_{x\to 1} f(x) = 1$$

$$\lim_{x\to\infty} f(x) = -1$$

$$\lim_{x\to -\infty} f(x) = -1$$



$$f'(x) = \frac{(x-1)^2(-2x+1) - (2(x-1)(-x^2+x+2))}{(x-1)^4} = \frac{(x^2-2x+1 \quad -2x+1 - (2x-1)(-x^2+x+2))}{(x-1)^4}$$

$$= \frac{x-5}{(x-1)^3}$$

$$0 = \frac{x-5}{(x-1)^3}$$

$$5 = x$$

C.P. $x=5$

$$f(5) = -\frac{9}{8}$$

	$(x-1)^3$	$x-5$	$f'(x)$
$(-\infty, 1)$	(−)	(−)	(+)
$(1, 5)$	(+)	(−)	(−)
$(5, \infty)$	(+)	(+)	(+)

x intercept $\quad 0 = \frac{2+x-x^2}{(x-1)^2}$

y intercept $\quad 0 = 2+x-x^2$

$-1, 2 = x$

$$f(x) = \frac{2+0-0}{(0-1)^2}$$

$$f(x) = 2$$

$$f''(x) = \frac{(x-1)^3 - (x-5)(3(x-1)^2)}{(x-1)^4}$$

$$f''(x) = \frac{2(7-x)}{(x-1)^4}$$

	$(x-1)^4$	$7-x$	$f''(x)$
$(-\infty, 1)$	(+)	(+)	(+)

$$(x-1)^4$$

$$f''(x) = \frac{2(7-x)}{(x-1)^4}$$

$$0 = \frac{2(7-x)}{(x-1)^4}$$

$$x = 7 \quad \text{I.P.}$$

$$f(7) = -\frac{10}{9}$$

	$(x-1)^4$	$7-x$	$f''(x)$
$(-\infty, 1)$	(+)	(+)	(+)
$(1, 7)$	(+)	(+)	(+)
$(7, \infty)$	(+)	(−)	(−)

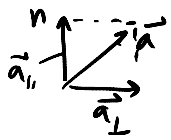
Examples

Thursday, October 19, 2006
9:04 AM

Ex1.

Given 2 vectors $\vec{a}, \hat{n}$

then $\vec{a}_{\parallel} = (\vec{a}\cdot\hat{n})\hat{n}$, $\vec{a}_{\perp} = \hat{n}\times(\vec{a}\times\hat{n})$



Show that ① $\vec{a}_{\parallel}\times\hat{n} = \vec{0}$ Parallel or antiparallel

② $\vec{a}_{\perp}\cdot\hat{n} = 0$ Perpendicular

③ $\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$ Components/ vector addition

$\vec{a} = [(\vec{a}\cdot\hat{n})\hat{n}] + [\hat{n}\times(\vec{a}\times\hat{n})]$

① $\vec{a}_{\parallel}\times\hat{n} = \vec{0}$

$(a\cdot\hat{n})\hat{n}\times\hat{n}$ if they point in same direction $\sin\measuredangle = 0$, $\hat{n}$ & $\hat{n}$ same direction. So ✓.

② $\hat{n}\times(\vec{a}\times\hat{n})\cdot\hat{n} = 0$ if they point in the same direction, the cross product = vector ⊥ to $\hat{n}$. Dot product of ⊥ vectors = 0. So ✓

vector ⊥ to $\hat{n}\cdot\hat{n}$

③ $\vec{a} = [(\vec{a}\cdot\hat{n})\hat{n}] + [\hat{n}\times(\vec{a}\times\hat{n})]$

$\vec{u}\times(\vec{v}\times\vec{w}) = \vec{v}(\vec{u}\cdot\vec{w}) - \vec{w}(\vec{u}\cdot\vec{v})$

$(\vec{a}\cdot\hat{n})\hat{n} + \hat{n}\times(\vec{a}\times\hat{n})$

$(\vec{a}\cdot\hat{n})\hat{n} + \vec{a}(\hat{n}\cdot\hat{n}) - \hat{n}(\hat{n}\cdot\vec{a})$

$\boxed{(\vec{a}\cdot\hat{n})\hat{n}} + \vec{a}(1) \boxed{-\hat{n}(\hat{n}\cdot\vec{a})}$

$= \vec{a}$ and that proves it.

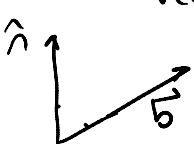
$\hat{n}\cdot\hat{n}$ are the same dot product of vector w/ itself = length. unit vector (b/c of $\wedge$) means it has length = 1.

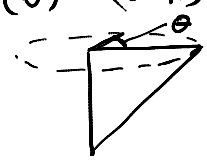
Ex.2 Show that if $\hat{n}$ is a unit vector & $\vec{b} \neq \vec{0}$ is a given vector that is not parallel/antiparallel to $\hat{n}$.

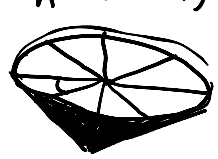
Then $\hat{n}$, $\vec{b}\times\hat{n}$ & $\hat{n}\times(\vec{b}\times\hat{n})$ are mutually orthogonal

$\vec{b} = (\vec{b}\cdot\hat{n})\hat{n} + \hat{n}\times(\vec{b}\times\hat{n})$ we just proved that ✓.

Rotating $\vec{b}$ by an angle θ around $\hat{n}$ yields a new vector $\vec{r}(\theta) = (\vec{b}\cdot\hat{n})\hat{n} + \hat{n}\times(\vec{b}\times\hat{n})\cos\theta + (\hat{n}\times\vec{b})\sin\theta$





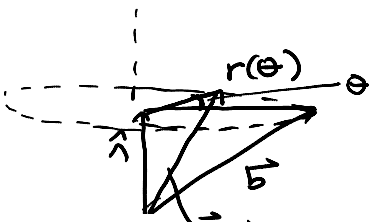


Mutually orthogonal means $\hat{n} \perp (\vec{b}\times\hat{n})$

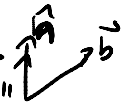
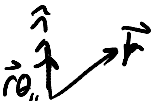
$\hat{n} \perp (\hat{n}\times(\vec{b}\times\hat{n}))$ So the 1st one

$\hat{n} \perp (\hat{n} \times (\vec{b} \times \hat{n}))$ 1st one works

$\vec{b} \times \hat{n} \perp (\hat{n} \times (\vec{b} \times \hat{n}))$



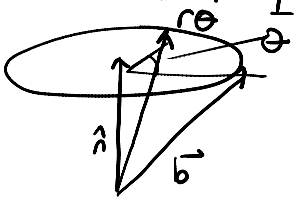
$\vec{b}_{\parallel} = \vec{r}\theta_{\parallel}$

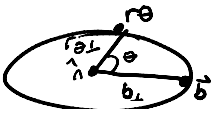
$\vec{r}(\theta)$ so if $\vec{b}_{\parallel} = \vec{b}_{\parallel}$  then $\vec{r}\theta_{\parallel} = \vec{r}\theta_{\parallel}$ 

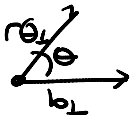
so it makes sense.

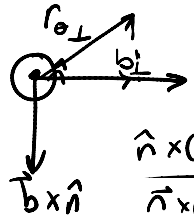
claim $\vec{b}_{\parallel} = \vec{r}\theta_{\parallel} = (\vec{b} \cdot \hat{n})\hat{n}$

what is $\vec{r}\theta_{\perp}$?









$$\frac{\hat{n} \times (\vec{b} \times \hat{n}) |\vec{b}_{\perp}| \cos\theta}{\hat{n} \times (\vec{b} \times \hat{n})}$$

$$\vec{r}(\theta)_{\perp} = \hat{n} \times (\vec{b} \times \hat{n}) \cos\theta$$

$$\vec{r}(\theta)_{\perp} = \hat{n} \times (\vec{b} \times \hat{n}) \cos\theta + |\vec{b}_{\perp}| \sin\theta \left(\frac{-\vec{b} \times \hat{n}}{|\vec{b} \times \hat{n}|} \right)$$

direction

$$\vec{b} \times \hat{n} = (\vec{b}_{\parallel} + \vec{b}_{\perp}) \times \hat{n} = \vec{b}_{\parallel} \times \hat{n} + \vec{b}_{\perp} \times \hat{n}$$
$$= \vec{b}_{\perp} \times \hat{n}$$
$$\vec{b} \times \hat{n} = \vec{b}_{\perp} \times \hat{n}$$
$$|\vec{b} \times \hat{n}| = |\vec{b}_{\perp} \times \hat{n}|$$
$$= |\vec{b}_{\perp}|$$

b/c $\vec{b}_{\perp}$ & $\hat{n}$ are $\perp$, & $\hat{n}$ is a unit vector

$$\vec{r}(\theta) = (\vec{b} \cdot \hat{n})\hat{n} + \hat{n} \times (\vec{b} \times \hat{n}) \cos\theta + (\hat{n} \times b) \sin\theta$$

And that proves it.

Projectile Motion, Orbits

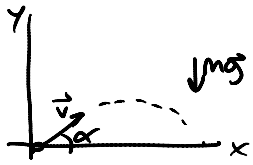
Friday, October 20, 2006
9:06 AM

Projectile Motion

$\vec{F} = m\vec{a}$

$\vec{a} = \frac{d\vec{v}}{dt} \quad \vec{v} = \frac{d\vec{r}}{dt}$

A particle is launched from the origin with speed v_0 at an ∡ α to the horizontal. Neglect air resistance. Determine projectile's trajectory.



We know what the force is. $\vec{F} = -mg\hat{j}$

Initial position at $t=0$

$\vec{r}(0) = \langle 0, 0\rangle$

Initial velocity at $t=0$

$|\vec{v}(0)| = v_0$

$\vec{v}(0) = \frac{d\vec{r}}{dt}(0) = v_0\langle\cos\alpha, \sin\alpha\rangle$

$m\frac{d\vec{v}}{dt} = \langle 0, -mg\rangle$ Motion is independent of this.

$\frac{d\vec{v}}{dt} = \langle 0, -g\rangle$

$\int_0^T d\vec{v} = \int_0^T \langle 0, -g\rangle dt$

$\vec{v}(t)\Big|_0^T = \vec{v}(T) - \vec{v}(0) = \langle 0, -gt\Big]_0^T\rangle$

$= \langle 0, -gT\rangle$

$\vec{a} = \langle 0, -g\rangle$

$\vec{v} = v_0\langle\cos\alpha, \sin\alpha\rangle + \langle 0, -gT\rangle$

$\vec{v} = \langle v_0\cos\alpha, v_0\sin\alpha - gT\rangle$ So this all works & everyone happy.

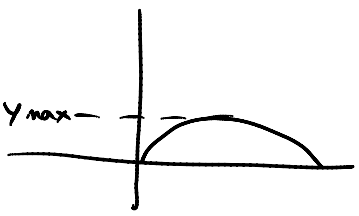
$\vec{v} = \frac{d\vec{r}}{dt}$

$\int_0^T \langle v_0\cos\alpha, v_0\sin\alpha - gT\rangle dt = \int_0^T \frac{d\vec{r}}{dt} dt$

$\left\langle v_0(\cos\alpha)T, v_0(\sin\alpha)T - \frac{gT^2}{2}\right\rangle = \vec{r}(T) - \vec{r}(0)$ $v_0\sin\alpha$ & $v_0\cos\alpha$ are just constants

$\langle 0,0\rangle + \left\langle v_0(\cos\alpha)T, v_0(\sin\alpha)T - \frac{gT^2}{2}\right\rangle = \vec{r}(T)$

So it looks like this:



$y = -c(x - x_0)^2 + y_{max}$

$-cx^2 + -2cxx_0 + x_0^2 + y_{max}$

$y = ax^2 + bx + c$

Projectile moves on a parabola

$$x(t) = [v_0(\cos\alpha)]t \qquad y(t) = [v_0(\sin\alpha)]t - \frac{gt^2}{2}$$

$$\frac{x}{v_0(\cos\alpha)} = t \qquad y(x) = [v_0\sin\alpha)]\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2$$

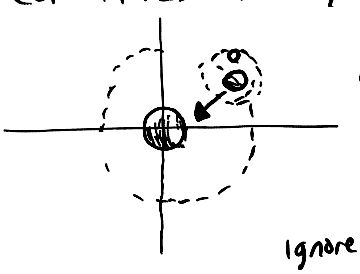
$$y(x) = \frac{x v_0 \sin\alpha}{v_0\cos\alpha} - \frac{g}{2}\frac{x^2}{v_0^2\cos^2\alpha}$$

$$y(x) = -\frac{1}{2}g\frac{x^2}{v_0^2\cos^2\alpha} + \frac{x\sin\alpha}{\cos\alpha}$$

1) When does the projectile hit the ground?
   $y(t)=0$ 2 solutions. $t=0$ & something else. the something else is the answer.
2) Where does it hit the ground?
   $x(t)$ you found above makes the position. or $y(x)=0$
3) What α gives us the maximum range?
   $\alpha = \frac{\pi}{4}$, but you find that by finding what α gives $y(t)$ max.
4) What is the maximum height achieved?
   $y(t)$ max is half the total time. Or at the max & α.
   $\frac{dy}{dt}=0$, C.P., etc.

Make Kerry's injury closer to her shoulder.

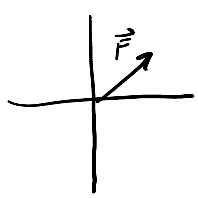
Central Forces & Planetary Motion



Central $\vec{F}$orce $= m\vec{a} = -f(r)\hat{r}$, r = distance to origin

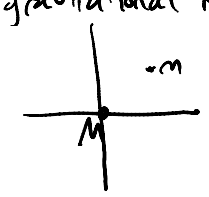
$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

ignore the moon... for now.



Specific example

gravitational force between masses m & M



$$\cancel{m}\vec{a} = \vec{F} = -\frac{GM\cancel{m}}{r^2}\hat{r}$$

$$\vec{a} = \frac{-GM}{r^2}\hat{r} \qquad G = 6.67E{-11}\frac{m^3}{kg\,s^2}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{-GM}{r^3}\vec{r} \qquad \vec{r} = \langle x, y, z\rangle$$

differential equation

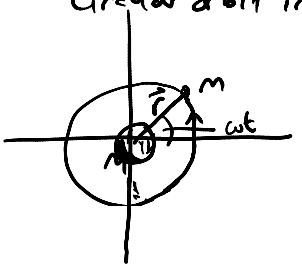
$$\left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right\rangle = -\frac{GM}{r^3}\vec{r}$$

$$= \frac{-GM}{(x^2+y^2+z^2)^{\frac{3}{2}}}\langle x, y, z\rangle$$

$|\vec{r}^3| \rightarrow$

Special Solution

Circular orbit in xy plane $z=0$



It would move around at uniform speed

- Guess the form:

$\vec{r} = \langle R\cos(\omega t), R\sin(\omega t) \rangle$, what is ω?

$|\vec{r}| = R$

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{R^3}\vec{r}$$

Central Force, Keplar

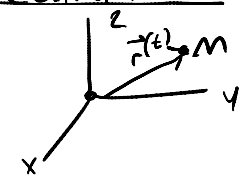
Monday, October 23, 2006
9:09 AM

2 facts: $\frac{d}{dt}(\vec{v}(t)\cdot\vec{w}(t)) = \frac{d\vec{v}}{dt}\cdot\vec{w}(t) + \vec{v}(t)\cdot\frac{d\vec{w}}{dt}$

$$\frac{d}{dt}(\vec{v}(t)\times\vec{w}(t)) = \underbrace{\frac{d\vec{v}}{dt}\times\vec{w}(t) + \vec{v}(t)\times\frac{d\vec{w}}{dt}}_{\text{order must be same.}}$$

These follow from product rule of differentiation

Central Force



m feels a force toward origin

X-FILES

$\vec{F} = -f(r)\hat{r}$ $\quad \hat{r} = \frac{\vec{r}}{|\vec{r}|}$ $\quad$ Strength of force depends on how far it is from origin.

$r = |\vec{r}|$

$\vec{F} = m\vec{a}$. $\quad m\cdot\frac{d\vec{v}}{dt} = m\vec{a}$

#1 A particle moving under a central force, conserves its angular momentum, $\vec{L}$.

$m\vec{a} = -f(r)\hat{r} = m\frac{d^2\vec{r}}{dt^2}$ ← trajectory $\vec{r}(t)$ satisfies this eqtn

$\vec{L} = \vec{r}\times\vec{p}$ $\quad$ Angular Momentum. Position × momentum $\quad$ $\vec{r}$ = vector to origin, Angular momentum about origin.

$\vec{p} = m\vec{v}$ $\quad$ Linear momentum

Angular momentum is constant $\quad$ $\vec{L}$ = constant vector

$\vec{L}(t) = \vec{L}(0)$

$\boxed{\frac{d}{dt}\vec{L} = 0}$ proves that ← need to show this.

THE TRUTH IS OUT THERE

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v}\times\vec{p}}_{\substack{0 \text{ b/c} \\ \text{linear momentum} \\ \text{points in same direction as velocity,} \\ \text{or } x\times y \text{ if } x \text{ \& } y \text{ are } \parallel \text{ is } 0.}} + \underbrace{\vec{r}\times\vec{F}}_{\text{torque } \vec{\tau}}$$

$p = mv$

$\frac{dp}{dt} = m\frac{dv}{dt}$

$F = m\frac{dv}{dt}$

LIKE VARIABLES

$$= 0 + \vec{r}\times\vec{F}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

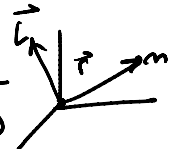
$\frac{d\vec{L}}{dt} = 0 = \vec{r}\times\vec{F}$ b/c $\vec{r}\parallel\vec{F}$ (antiparallel)

$\frac{dL}{dt} = 0 = r \times F$ b/c $r \parallel F$ (anti parallel)

Yay, we proved it!

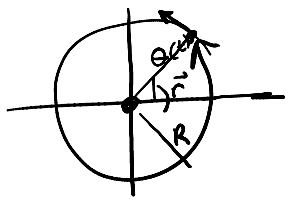
<u>#2</u> Particle moves in a plane $\perp$ to $\vec{L}$ & hence its motion is restricted to 2 dimensions.

$\vec{L} = \vec{r} \times p$, $\underbrace{\begin{matrix} \vec{r} \cdot \vec{L} = 0 \\ p \cdot \vec{L} = 0 \end{matrix}}_{\text{motion is in the plane } \perp \text{ to } \vec{L}}$ b/c $\vec{r}$ & $\vec{L}$ are $\perp$ the $\vec{r}$ is always $\perp$ to $\vec{L}$. position $\perp$ to angular momentum or it doesn't move in a circle or side-to-side.



<u>#3</u>

$\vec{F} = -f(r)\hat{r} = m\frac{d^2\vec{r}}{dt^2}$. Find special soln corresponding to circular orbit.



$\vec{L}$ points in $\hat{z}$ direction, out of the page.

$\odot \vec{L} = L_z \hat{k}$

$\vec{L}$ is in $\hat{k}$ direction

$\vec{r} = R\langle \cos[\theta(t)], \sin[\theta(t)] \rangle$

Constant $= R^2 = \vec{r} \cdot \vec{r}$ $\qquad \vec{r} \cdot \vec{r} = |\vec{r}|^2$

$0 = \frac{d}{dt}(R^2) = \frac{d}{dt}(\vec{r} \cdot \vec{r})$

$= \frac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \frac{d\vec{r}}{dt}$ order doesn't matter

$0 = 2\vec{r} \cdot \vec{v}$ if the dot product is 0, we know one is $\perp$.

$\frac{d}{dt}0 = \frac{d}{dt}(\vec{r} \cdot \vec{v})$ constant doesn't matter. $\vec{r}$ is $\perp$ $\vec{v}$

$0 = \frac{d\vec{v}}{dt} \cdot \vec{v} + r \cdot \frac{d\vec{v}}{dt}$

$0 = \underbrace{\vec{v} \cdot \vec{v}}_{v^2} + \vec{r} \cdot \vec{a}$ $\qquad \dot{\theta} = \frac{d\theta}{dt}$

$-(\vec{r} \cdot \vec{a}) = v^2$

$\frac{d\vec{r}}{dt} = R\langle -\sin(\theta t) \cdot \dot{\theta}, \cos(\theta t) \cdot \dot{\theta} \rangle$

$\vec{v} = R\dot{\theta}\langle -\sin\theta, \cos\theta \rangle$

$-\vec{r} \cdot \vec{a} = |\vec{v}|^2$

$\vec{r} = R\langle \cos\theta, \sin\theta \rangle, \theta = \theta(t)$

$\vec{a} = -\frac{f(r)\hat{r}}{m} = \frac{-f(R)}{m}\langle \cos\theta, \sin\theta \rangle$ $\quad r = |\vec{r}| = R$

$-\vec{r} \cdot \vec{a} = R^2\dot{\theta}^2[\sin^2\theta + \cos^2\theta]$

$-\vec{r} \cdot \vec{a} = R^2\dot{\theta}^2$

$R\langle \cos\theta, \sin\theta \rangle \cdot -\frac{f(R)}{m}\langle \cos\theta, \sin\theta \rangle = R^2\dot{\theta}^2$

$R\left(\frac{-f(R)}{m}\right)[(\cos^2\theta - \sin^2\theta)] = R^2\dot{\theta}^2$

$\frac{f(R)}{mR} = \dot{\theta}^2$

$\sqrt{\frac{f(R)}{mR}} = \dot{\theta}$ [illegible]

$$\sqrt{\frac{f(R)}{mR}} = \dot{\theta} = \text{constant angular } \omega \text{ speed}$$

$$\int \sqrt{\frac{f(R)}{mR}} = \int \dot{\theta} = \int \dot{\theta}\, dt = \int \omega\, dt$$

$$\theta(t) = \omega t + C \qquad \theta(0) = \theta_0$$

$$\theta(t) = \omega t + \theta_0$$

$$\vec{r}(t) = R\langle \cos(\omega t + \theta_0), \sin(\omega t + \theta_0)\rangle$$

$$\omega = \sqrt{\frac{f(R)}{mR}} \qquad \text{circular trajectory}$$

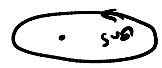
<u>Kepler's Laws</u>

Kepler 1571-1630

Newton's Laws published 1687

1. Planets move in elliptical orbits with the sun at one focus

   in circular case, that means sun at origin b/c circular case = bring 2 foci together.



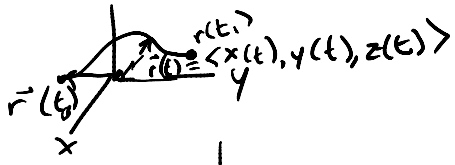
2. Area swept out by radius vector from the sun in equal times are equal.

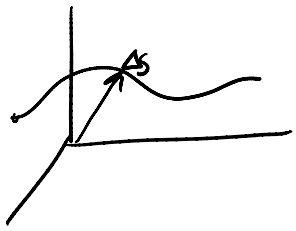
angular speed is constant

Arc Length

Tuesday, October 24, 2006
8:59 AM

Arclength of a curve in 3D





$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \qquad r(t) \quad t_0 \le t \le t_1$$

$$\Delta x \approx \frac{\Delta x}{\Delta t}\Delta t$$

$$\Delta s \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\ \Delta t$$

Add all Δs letting $\Delta t \to 0$

$$\int \Delta s \approx \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

(arc length) $$S = \int_{t_0}^{t_1} \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{|\vec{v}|}\ dt$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

so $$\boxed{S = \int_{t_0}^{t_1} |\vec{v}|\, dt}$$

ex. 1

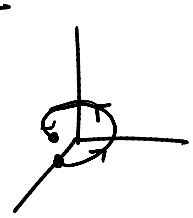
$$\vec{r} = \langle a\cos(\omega t),\ a\sin(\omega t),\ bt \rangle$$

$$0 \le t \le \underbrace{\frac{2\pi}{\omega}}_{}$$

this tells you that you go around one revolution.

Makes a helix b/c x & y make a circle, but bt changes magnitude.

increasing z position



$L = S$

$$l = \int_{t_0}^{t_1} |\vec{v}|\, dt$$

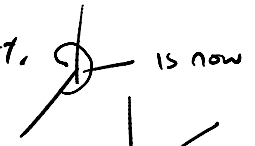
$$\vec{v} = \langle -a\omega\sin(\omega t),\ a\omega\cos(\omega t),\ b \rangle$$

$$L = \int_{t_0}^{t_1} \sqrt{a^2\omega^2\sin^2(\omega t) + a^2\omega^2\cos^2(\omega t) + b^2}$$

$$L = \int_{t_0}^{t_1} \sqrt{a^2\omega^2 + b^2}\ dt$$

$$L = \left[\sqrt{a^2\omega^2 + b^2}\, t\right]_{t_0 = \omega}^{t_1 = 2\pi}$$

$$L = \sqrt{a^2\omega^2 + b^2}\,\frac{2\pi}{\omega}$$

<u>Check</u> Take the line of wire & straighten it out.  is now

circumfrence of ⊙ is $2\pi r$

$2\pi a$

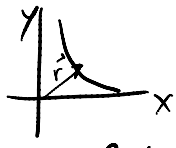
rise in z direction $= b\left(\frac{2\pi}{\omega}\right)$ b/c of bt.

$$\sqrt{(2\pi a)^2 + \left(b\frac{2\pi}{\omega}\right)^2} = \text{distance/length of wire}$$

$$\sqrt{4\pi^2 a^2 + \frac{4b^2\pi^2}{\omega^2}}$$

$$L = \sqrt{a^2\omega^2 + b^2}$$

<u>ex 2.</u> Calculate length of the curve given implicitly $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ in 1st quadrant



$\vec{r}(t) = \langle x(t), y(t)\rangle$

$x = \sin^3 t \quad y = \cos^3 t$

<u>Check</u>

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

$$(\sin^3 t)^{\frac{2}{3}} + (\cos^3 t)^{\frac{2}{3}} = 1$$

$$\sin^2 t + \cos^2 t = 1$$

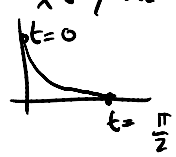
$$1 = 1$$

TRIG SUBSTITUTION

$0 \le t \le \frac{\pi}{2}$

<u>Check</u>

x & y are both pos



$\vec{r} = \langle \sin^3 t, \cos^3 t\rangle$

$\vec{v} = \langle 3\sin^2 t\cos t, 3\cos^2 t(-\sin t)\rangle$

$$\vec{v} = \int_0^{\frac{\pi}{2}} \underbrace{\sqrt{(3\sin^2 t\cos t)^2 + (3\cos^2 t(-\sin t))^2}}_{\sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}}$$

$$L = \int_0^{\frac{\pi}{2}} 3\sin t\cos t\, dt$$

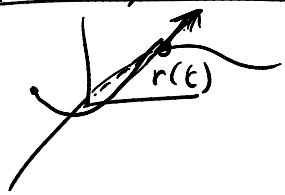
$$L = 3\left.\frac{\sin^2 t}{2}\right]_0^{\frac{\pi}{2}}$$

$L = \frac{3}{2}$ is the arc length.

Amidst oceans crashing against the shore, moonlit waters and a dead boat, a house hovering

A house hovering above it all, and a baby is born...

Unit tangent vector (to a parametric curve)



$\frac{d\vec{r}}{dt} = \vec{v}$ is tangent to the curve.

$\hat{T} = \frac{\vec{v}(t)}{|\vec{v}|}$ unit vector w/ length one that gives you direction.

velocity = speed times unit vector.

$\vec{v} = |\vec{v}|\hat{T}$

Characterization of curve

the rate that $\hat{T}$ is changing as we move along the curve.

We don't want $\frac{d\hat{T}}{dt}$ b/c then our answer depends on how we parameterized.

$\frac{d\hat{T}}{ds}$ where s is arc length is better.

Calculate instead, $\frac{d\hat{T}}{ds}$, rate of change of $\hat{T}$ w/ respect to s.

$$\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt}\frac{dt}{ds} = \frac{1}{ds}dt\ \frac{d\hat{T}}{dt}\left(\frac{ds}{dt}\right)^{-1}$$

$|\vec{v}|$ inherent to the curve & independent of parameterization

$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$ $\quad \vec{v} = \frac{dr}{dt}$

$$\frac{d\hat{T}}{ds} = \frac{1}{|\vec{v}|}\frac{d\hat{T}}{dt} = \underset{\text{magnitude}}{\kappa}\ \underset{\text{direction.}}{\hat{N}}$$

This defines κ & $\hat{N}$.

κ = curvature $= \left|\frac{d\hat{T}}{ds}\right|$

$\hat{N}$ = principal normal to curve.

normal means $\perp$. $= \frac{\frac{d\hat{T}}{ds}}{\left|\frac{d\hat{T}}{ds}\right|}$

Show that $\hat{N} \perp \hat{T}$

$$\frac{d}{dt}(\hat{T}\cdot\hat{T}) = \frac{d}{dt}(\hat{T}^2) = \frac{d}{dt}(1) = 0$$

$$\frac{d\hat{T}}{dt}\hat{T} + \hat{T}\frac{d\hat{T}}{dt}$$

$$2\hat{T}\frac{d\hat{T}}{dt}$$

$$2(|\vec{v}|\kappa\hat{N})\cdot\hat{T}$$

$$\Rightarrow \hat{N}\cdot\hat{T} = 0 \Rightarrow \hat{N} \perp \hat{T}$$

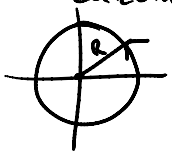
$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\hat{T}}{ds} = \frac{1}{|\vec{v}|}\frac{d\hat{T}}{dt} = \kappa\hat{N}$$

<u>ex1.</u>

$\vec{r} = \langle R\cos t, R\sin t, 0\rangle$

calculate curvature & principal normal



$$\frac{d\vec{r}}{dt} = \vec{v} = \langle -R\sin t, R\cos t, 0\rangle$$

$$|\vec{v}| = \sqrt{R^2\sin^2 t + R^2\cos^2 t} = \sqrt{R^2} = R$$

$$|\vec{v}| = R$$

$$\hat{T} = \langle -\sin t, \cos t, 0\rangle \quad \text{b/c } \hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\kappa\cdot\hat{N} = \frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt}\frac{1}{|\vec{v}|} = \frac{\langle -\cos t, -\sin t, 0\rangle}{R}$$

$$\hat{N} = \langle \cos t, \sin t, 0\rangle$$

$$\kappa = \frac{1}{R}$$

<u>To prove:</u>

$$\vec{a}\times\vec{b} = \vec{b}^{\,3}\kappa\hat{N}\times\hat{T}$$

where $\vec{v} = \frac{d\vec{r}}{dt}$ & $\vec{a} = \frac{d\vec{v}}{dt}$

$$\Rightarrow \kappa = \frac{|\vec{a}\times\vec{v}|}{|\vec{v}|^3}$$

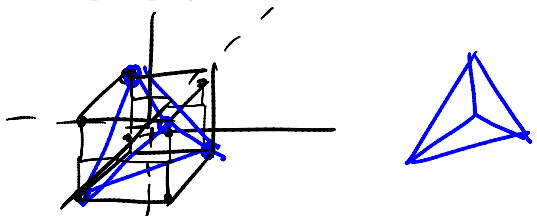
Calculus HW

Wednesday, October 25, 2006
8:59 AM

252-1 Calculus **Homework #4** **Professor Silber**

Problems A1 through A5 are due Monday, October 30 at the beginning of class. The second exam will be Wednesday, November 1 in class.

A1. Consider a cube with eight vertices at $(\pm 1, \pm 1, \pm 1)$; in the previous problem set we showed that four of these vertices can be connected to form the edges of a tertahedron (a triangular pyramid with all edges equal). Compute the volume and surface area of this tetrahedron.



A2. Consider the three points $P = (2, 1, 1)$, $Q = (4, 2, 1)$, & $R = (1, 1, 2)$

(a) Find the cosine of the angle between $\vec{PQ}$ and $\vec{PR}$.

(b) Find the parametric equation of a line through P and Q.

(c) Find the equation of the plane containing P, Q, and R.

(d) Find the area of the triangle with vertices P, Q, and R.

(e) Find the volume of the tetrahedron with vertices P, Q, R and the origin.

(f) Find the distance between the origin and the plane containing P, Q, and R. Hint: this can be done easily using parts d and e.)

A3. Suppose $\vec{U} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{V} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Find two vectors, $\vec{A}$ and $\vec{B}$, such that

$$\vec{A} + \vec{B} = \vec{U} \qquad \vec{A} \cdot \vec{V} = 0 \qquad \vec{B} \times \vec{V} = 0 \ .$$

A4. Consider the curve defined by the vector

$$\vec{r}(t) = e^t \cos(t)\ \mathbf{i} + e^t \sin(t)\ \mathbf{j} \qquad 0 \le t \le 2\pi$$

(a) Graph the curve and compute its arclength.

(b) Compute the unit tangent vector $\hat{T}$ for the curve.

(c) Show that the angle between a line from the origin to a point on the curve and the tangent line to the curve is constant and compute the angle.

A5. A roller coaster has height y (in meters) as a function of horizontal distance x (in meters) given by:

$$y = 90 \sin\left(\frac{x}{90}\right) \qquad 0 \le x \le 90\pi$$

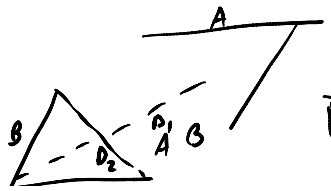
(a) Graph the roller coaster and find where the height is a maximum. Compute the curvature of the roller coaster.

(b) Suppose a car of mass M on the roller coaster is moving with speed V. Gravity pulls down on the car with a force of Mg where $g \approx 10$ meters/(sec)^2. How fast does the car need to be moving when it reaches the maximum height such that a passenger will feel a moment of weightlessness? What happens if the car is moving faster?

Practice probs & Tetrahedron

Wednesday, October 25, 2006
9:00 AM

A6.



$\vec{D}_1 = \vec{A} - \vec{B}$

Prove that $|\vec{D}_1|^2 + |\vec{D}_2|^2 = 2|\vec{A}|^2 + 2|\vec{B}|^2$

$|D_2|^2 + |D_1|^2 = |\vec{A} + \vec{B}|^2 + |\vec{A} - \vec{B}|^2$

$\times$ not true $= |\vec{A}|^2 + 2|\vec{A}||\vec{B}|$

$\vec{v} \ni |\vec{v}|^2 = \vec{v} \cdot \vec{v}$

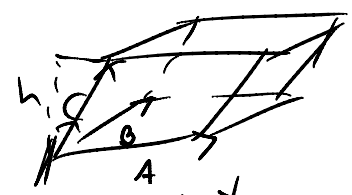
$= (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) + (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B})$

$= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}$

$= 2(\vec{A} \cdot \vec{A}) + 2(\vec{B} \cdot \vec{B})$

$= 2|\vec{A}|^2 + 2|\vec{B}|^2$

$\therefore$ Sum of squares of lengths of diagonals is equal to the sum of the squares of the lengths of the sides.



θ $|h| = |\vec{C}| \cos\theta$

$\tilde{h} = \vec{A} \times \vec{B}$ not assuming anything about $|\tilde{h}|$

$\tilde{h} \cdot \vec{C} = |\tilde{h}| |\vec{C}| \cos\theta$

$(\vec{A} \times \vec{B}) \cdot \vec{C} = |\vec{A} \times \vec{B}| |\vec{C}| \cos\theta$

$\frac{(\vec{A} \times \vec{B}) \cdot \vec{C}}{|\vec{A} \times \vec{B}| |\vec{C}|} = \cos\theta$

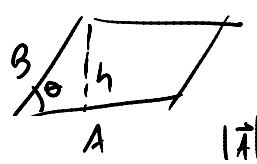
$|h| = |\vec{C}| \left(\frac{(\vec{A} \times \vec{B}) \cdot \vec{C}}{|\vec{A} \times \vec{B}| |\vec{C}|} \right) = \frac{(\vec{A} \times \vec{B}) \cdot \vec{C}}{|\vec{A} \times \vec{B}|}$ this is height

we need area of base

$A_{\square} = |\vec{A} \times \vec{B}|$ $\vec{A} \times \vec{B}$ is area of base, of a $\square$.

$V = |\vec{A} \times \vec{B}| \left(\frac{(\vec{A} \times \vec{B}) \cdot \vec{C}}{|\vec{A} \times \vec{B}|} \right)$

$V = (\vec{A} \times \vec{B}) \cdot \vec{C}$



$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n}$

$|\vec{A}||\vec{B}| \sin\theta$

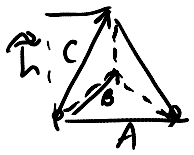
$\downarrow$

Area of parallelogram

Tetrahedron

Tetrahedron

$V_{tet} = \frac{1}{3} h A$



A triangle is $\frac{1}{2}$ ▱

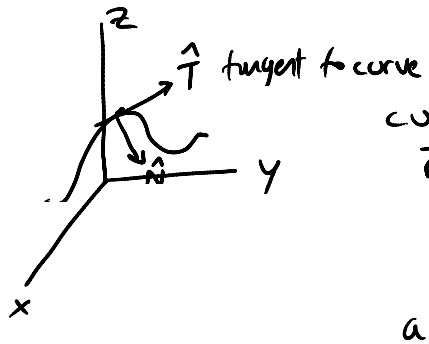
$\frac{1}{2}|\vec{A} \times \vec{B}|$ = area of △

$$V = \left(\left[\frac{1}{2} |\vec{A} \times \vec{B}| \right] \frac{(\vec{A} \times \vec{B}) \cdot \vec{C}}{|\vec{A} \times \vec{B}|} \right) \frac{1}{3}$$

Area of parallelogram

Tangent to curve, Kepler

Wednesday, October 25, 2006
9:26 AM



curve in 3D given parametrically:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$a = \frac{d\vec{v}}{dt} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right\rangle$$

$\hat{T}$ is unit tangent vector
$\hat{N}$ is the principal normal
K is the curvature

$\hat{T}$ points in direction of $\vec{v}$.

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

S is arclength

$$\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt}\frac{dt}{ds} = \frac{d\hat{T}}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{d\hat{T}}{dt}\left(\frac{1}{|\vec{v}|}\right)$$

Define K + $\hat{N}$ by $\frac{d\hat{T}}{ds} = K\hat{N}$ direction is principal normal, K, curvature, is length.

$$K\hat{N} = \frac{d\hat{T}}{dt}\left(\frac{1}{|\vec{v}|}\right) = \frac{d\hat{T}}{ds}$$

$\hat{N} \perp \hat{T}$ (we showed this last time)

Derive alternative expression for K in terms of $\vec{a}$ & $\vec{v}$.

comes from $\hat{T} = \frac{\vec{v}}{|\vec{v}|}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(|\vec{v}|\hat{T})$$

Product rule

$$= \frac{d|\vec{v}|}{dt}\hat{T} + \underbrace{\frac{d\hat{T}}{dt}}_{|\vec{v}|K\hat{N}}|\vec{v}|$$

b/c $\frac{d\hat{T}}{dt}\frac{1}{|\vec{v}|} = K\hat{N}$

$$= \frac{d|\vec{v}|}{dt}\hat{T} + |\vec{v}|^2 K\hat{N}$$

we want to extract these other terms

$$\vec{a} = \frac{d|\vec{v}|}{dt}\hat{T} + |\vec{v}|^2 K\hat{N}$$

$$\vec{a}\times\vec{v} = \left[\left(\frac{d|\vec{v}|}{dt}\underbrace{\hat{T}\times\vec{v}}_{\text{point in same direction, so } 0}\right)\right] + \left[\left(|\vec{v}|^2 K\hat{N}\right)\times\vec{v}\right]$$

$$\vec{a}\times\vec{v} = \left[\left(|\vec{v}|^2 K\hat{N}\right)\times\vec{v}\right] \qquad \vec{v} = |\vec{v}|\hat{T}$$

$\vec{a} \times \vec{v} = \left[(|\vec{v}|^2 \kappa \hat{N}) \times \vec{v} \right]$ $\qquad \vec{v} = |\vec{v}| \hat{T}$

$\vec{a} \times \vec{v} = (|\vec{v}|^2 \kappa \hat{N}) \times (|\vec{v}| \hat{T})$ $\qquad$ you can't divide by a vector.

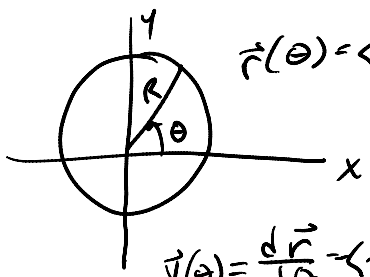
$\vec{a} \times \vec{v} = |\vec{v}|^3 \kappa (\hat{N} \times \hat{T})$ $\qquad$ distributive cross product

$|\vec{a} \times \vec{v}| = |\vec{v}|^3 \kappa |\hat{N} \times \hat{T}|$ $\qquad$ now we have lengths

$|\vec{a} \times \vec{v}| = |\vec{v}|^3 \kappa |\hat{N}||\hat{T}| \sin\theta$ $\qquad$ since $\hat{N} \perp \hat{T}$, $\sin = 90 = 1$ and $\hat{N}$ & $\hat{T}$ are unit vectors so they're both 1. $|\hat{N}||\hat{T}| = 1$

$\dfrac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3} = \kappa$ curvature!

in terms of a circle, the curvature is $\frac{1}{R}$. If you have each pt. on a curve & you're trying to fit a circle, the curvature means how many circles you'd have to fit. One really big one means low curvature, tiny ones mean large curvature.



$\vec{r}(\theta) = \langle R\cos\theta, R\sin\theta, 0 \rangle$

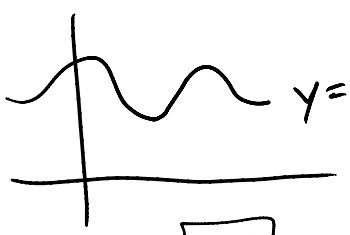
$\vec{v}(\theta) = \dfrac{d\vec{r}}{d\theta} = \langle -R\sin\theta, R\cos\theta, 0 \rangle$

$\vec{a}(\theta) = \dfrac{d\vec{v}}{d\theta} = \langle -R\cos\theta, -R\sin\theta, 0 \rangle$

$\vec{a} \cdot \vec{v} = R^2 \cos\theta \sin\theta + (-R^2 \cos\theta \sin\theta) = 0$ $\qquad \vec{a}$ & $\vec{v}$ are $\perp$.

$\vec{a} \perp \vec{v} = |\vec{a} \times \vec{v}| = |\vec{a}||\vec{v}|$

$$\boxed{\kappa = \frac{|\vec{a}||\vec{v}|}{|\vec{v}|^3}} = \frac{R \cdot R}{R^3} = \frac{1}{R}$$ Curvature $= \frac{1}{R}$.



$y = f(x)$

Calculate curvature in every pt. on the curve.

$\vec{r}(t) = \langle x(t), y(t), 0 \rangle$

let $\boxed{x = t}$

$y = f(t) \longrightarrow \vec{r}(t) = \langle t, f(t), 0 \rangle$

$\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \langle 1, f'(t), 0 \rangle$

tangent vector to the curve, velocity

$\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \langle 0, f''(t), 0 \rangle$

acceleration.

$$\kappa = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}$$

$|\vec{v}| = \sqrt{1 + f'(t)^2}$

$\vec{a} \times \vec{v} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & f'' & 0 \\ 1 & f' & 0 \end{vmatrix}$

$\hat{i}0 - \hat{j}0 + (0 - f'')$

$-f''$

$$\kappa = \frac{|f''(t)|}{|\vec{v}|^3}$$ absolute value, so no neg

$$\boxed{\kappa = \frac{|f''(t)|}{\left|\sqrt{1 + f'(t)^2}\right|^3}}$$ if $y = f(x)$

$$\boxed{\sqrt{1+f'(t)^2}}^3$$

replace t by x

$$K = \frac{|f''(x)|}{(1+f'(x)^2)^{\frac{3}{2}}}$$

expect K is a max at $x=0$
$K \to 0$ as $x \to \pm\infty$

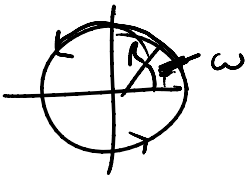
$y = x^2$
$y' = 2x$
$y'' = 2$

$$K = \frac{2}{(1+4x^2)^{\frac{3}{2}}}$$

K will be largest when $x=0$. Smallest denominator can be is 1, & that happens at $x=0$ $K=2$, $x=0$

$x \to \pm\infty$, denominator gets larger
$K \to 0$

$$\vec{F} = -f(|\vec{r}|)\,\hat{r} - m\frac{d^2\vec{r}}{dt^2} \qquad \vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t), 0\rangle$$



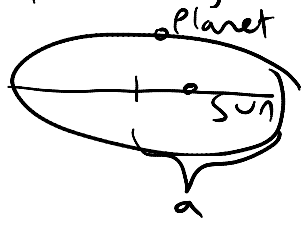
$$\omega^2 = \frac{f(R)}{mR}$$

Can get this by plugging circular solution into newton's equation. $\vec{F} = m\vec{a}$

$|\vec{r}| = R$ b/c it's circular

<u>Last of Kepler's Laws</u>

Third on Kepler's Laws of Planetary motion
Square of the period of revolution is proportional to the cube of the semi major axis of the ellipse



T = period of orbit

semi major axis of an ellipse is a

$$T^2 = C a^3 \qquad C \text{ is a constant}$$

Use it in special case of circular orbit. Circular case, a is the same as radius of the circle

$$(T)^2 = \left(\frac{2\pi}{\omega}\right)^2$$

$$T^2 = \frac{(2\pi)^2}{} mR = CR^3$$

$$f(R) = \frac{\overset{\omega^2}{(2\pi)^2 m}}{\underbrace{C}_{\substack{\text{new constant} \\ K}}} \frac{1}{R^2}$$

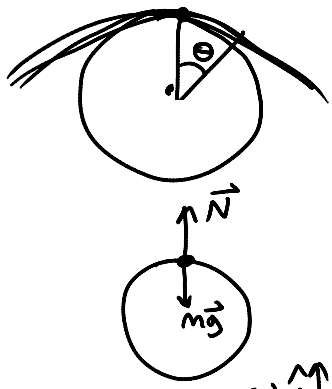
$$\vec{F} = -f(R)\hat{r}$$

$$= -\frac{K}{R^2}\hat{r} \qquad K = GMm$$

$$= -\frac{GMm}{R^2}\hat{r}$$

Particle slides off sphere

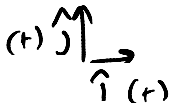
Friday, October 27, 2006
9:04 AM



Particle slides off a sphere.
Find the ∡ θ where it first leaves the sphere. Neglect friction. No rolling

$\vec{F} = m\vec{a}$ ← normal force

$F = -mg\hat{j} + f_n\hat{r}$



The point at which it's just about to lose contact w/ the sphere, $f_n \to 0$
Figure out at what ∡ $f_n = 0$

$\hat{r}$ is a unit vector pointing radially outward in the direction of the particle.

f_n depends on θ → $f_n = f_n(\theta)$

$f_n(\theta) > 0$ for $\theta = \theta^*$

$f_n(\theta) = 0$ for $0 \geq \theta^*$

Goal is to find $f_n(\theta)$, then $f_n(\theta^*) = 0$ determines θ^*

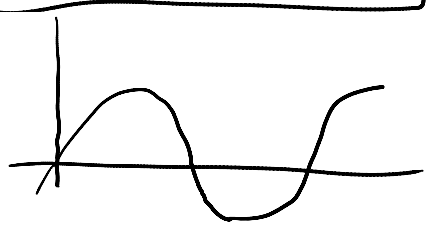
$\vec{a} = \frac{d\vec{v}}{dt}$ $\quad \vec{v} = |\vec{v}|\hat{T}$

$$\frac{d\vec{v}}{dt} = \frac{d|\vec{v}|}{dt}\hat{T} + |\vec{v}|\frac{d\hat{T}}{dt}$$ ← relate to the curvature

$$\kappa\hat{N} = \frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}$$

$$\frac{d\hat{T}}{dt} = \underbrace{\frac{ds}{dt}}_{\text{speed}}\kappa\hat{N}$$

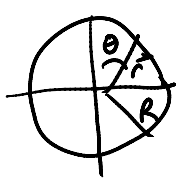
$$\boxed{m\vec{a} = m\frac{d|\vec{v}|}{dt}\hat{T} + m|\vec{v}|^2\kappa\hat{N}}$$ Completely general formula. Come from def. of acceleration.



sine wave

If you have constant speed, the first term is 0. Meaning $m\vec{a} = m|\vec{v}|^2\kappa\hat{N}$

For circle



$\kappa = \frac{1}{R}$ $\quad \hat{N} = -\hat{r}$

principal normal for a circle is always inward, $\hat{r}$, while $\hat{N}$ is

$$m\frac{d|\vec{v}|}{dt}\hat{T} + m|\vec{v}|^2 \kappa \hat{N} = -mg\hat{j} + f_n\hat{r}$$ outward.

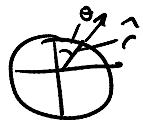
$$m\frac{d|\vec{v}|}{dt}\hat{T} - m|\vec{v}|^2\frac{1}{R}\hat{r} = -mg\hat{j} + f_n\hat{r}$$

You can't solve vectors conventionally. You can only do that w/ magnitudes.

$$\hat{r}\cdot\left[m\frac{d\vec{v}}{dt}\hat{T} - m|\vec{v}|^2\frac{1}{R}\hat{r}\right] = \left[-mg\hat{j} + f_n\hat{r}\right]\cdot\hat{r}$$

$\hat{r}\perp\hat{T}$, $\hat{r}=1$ so $\hat{r}\cdot\hat{r} = |\hat{r}| = 1$

$$-\frac{m|\vec{v}|^2}{R} = -mg(\hat{j}\cdot\hat{r}) + f_n$$



$$\hat{r} = \langle \sin\theta, \underbrace{\cos\theta}_{\hat{j}\cdot\hat{r}} \rangle$$

$$-\frac{m|\vec{v}|^2}{R} = -mg\cos\theta + f_n$$

$$-\frac{m|\vec{v}|^2}{R} + mg\cos\theta = f_n$$

Conservation of Energy theorem

$$E = \frac{1}{2}mv^2 + mgh$$

initial energy = energy at $\angle\theta$

Kinetic + Potential

$$mgR = \frac{1}{2}m|\vec{v}|^2 + mgR\cos\theta$$

$$2(mgR - mgR\cos\theta) = m|\vec{v}|^2$$

$$2mgR(1-\cos\theta) = m|\vec{v}|^2$$

$$f_n = -\frac{2mgR(1-\cos\theta)}{R} + mg\cos\theta$$

$$f_n = mg[3\cos\theta - 2]$$

R doesn't come into the formula

$$m\vec{a} = m\frac{d|\vec{v}|}{dt}\hat{T} + m|\vec{v}|^2\kappa\hat{N} = -mg\hat{j} + f_n\hat{r}$$

$f_n = 0$ at $\theta = \theta^*$

$$0 = mg[3\cos\theta - 2]$$

$$\cos\theta = \frac{2}{3}$$

$$m\frac{d\vec{v}}{dt} = -mg\hat{j} + f_n\hat{r}$$

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = -mg\hat{j}\cdot\vec{v} + f_n\underbrace{\hat{r}\cdot\vec{v}}_{0}$$

0 b/c $\vec{v}$ is always $\perp$ to $\hat{r}$.

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = -mg\hat{j}\cdot\vec{v}$$

$$\vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = -mg\hat{j}\cdot\frac{dx}{dt}\frac{dy}{dt} \qquad \hat{j}\cdot\vec{v} = \frac{dy}{dt}$$

$$\int m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = -mg\int\frac{dy}{\ldots}$$

$$\int m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = -mg\int\frac{dy}{dt}\,dt$$

$$m\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\frac{d}{dt}|\vec{v}|^2$$

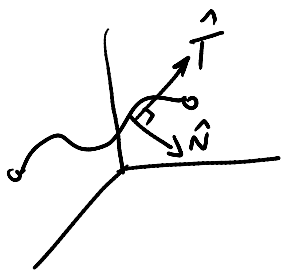
$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d\vec{v}}{dt}\vec{v} + \vec{v}\frac{d\vec{v}}{dt}$$

$$\frac{1}{2}m\frac{d\vec{v}}{dt}\vec{v} + \frac{1}{2}m\frac{d\vec{v}}{dt}\vec{v} = m\vec{v}\frac{d\vec{v}}{dt}$$ so that's why that's =.

$$\int_0^T \frac{1}{2}m\frac{d}{dt}(|\vec{v}|^2)\,dt = -mg\int_0^T\frac{dy}{dt}\,dt$$

$$\frac{1}{2}m|\vec{v}|^2\Big]_0^T = -mgy\Big]_0^T$$

$$\frac{1}{2}m|\vec{v}_f|^2 + mgy_f = \frac{1}{2}m|\vec{v}_0|^2 + mgy_0$$ Conservation of energy formula.



$$K = \frac{|\vec{a}\times\vec{v}|}{|\vec{v}|^3}$$

Conic Sections

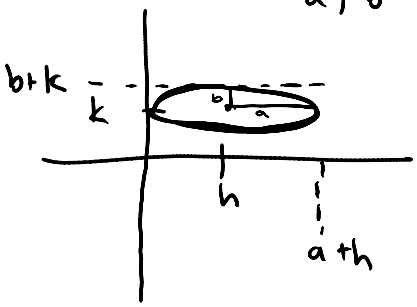
Monday, October 30, 2006
10:03 AM

Conic Sections Review

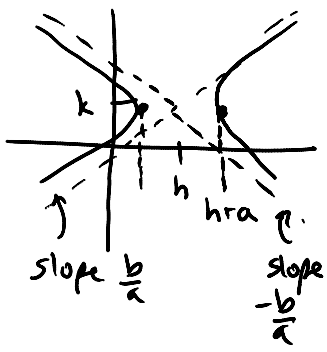
Ellipse, Parabola, & Hyperbola. Conic sections = Cone.

Ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ This ellipse is $\parallel$ to x & y axes. We could rotate it.

(h, k) center of ellipse
a, b semimajor & semiminor axes.



Hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ This hyperbola is not rotated.



$(h+a, k)$
$(h-a, k)$ 2 vertices.

$\frac{(h+a-h)^2}{a^2} - \frac{(k-k)^2}{b^2} = 1$

$1 = 1$ ✓ So the x coordinates are $(h+a, k)$ $(h-a, k)$

Equation for asymptotes
$\pm \frac{b}{a} = \frac{(y-k)}{(x-h)}$ take any point (x, y) and find slope w/ (h, k)

Factor equation

$\left(\frac{x-h}{a} - \frac{y-k}{b}\right)\left(\frac{x-h}{a} + \frac{y-k}{b}\right) = 1$

$\frac{x-h}{a} - \frac{(y-k)}{b} \to 0$ $(x, y) \to \infty$ (look at graph)

b/c we need product to be 1

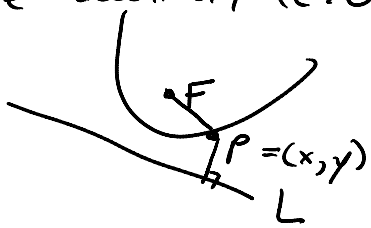
One of 2 factors goes to 0. The other goes to infinity.

$\frac{a}{a} - \frac{b}{b} = 0$ (b/c of slope → look above)

$$\left(\frac{x-h}{a}\right) = \pm\left(\frac{y-k}{b}\right)$$

Focus directrix ~~of conic sections~~

F = focus = point in (x,y) plane
L = directrix = line in (x,y) plane } F not in L
e = eccentricity $(e > 0)$



$$e = \frac{|PF|}{|PL|}$$

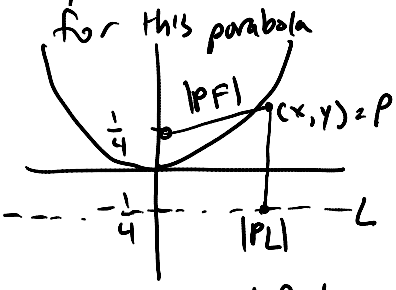
$e = 1$
this means Parabola. The distance from F to P is always equal to distance from P to L.

$e > 1$
hyperbola

$0 < e < 1$
this means distance from F to P is shorter than P to L
So ellipse.

<u>Example</u> parabola Focus $= (0, \frac{1}{4})$ Parabola, so $e = 1$
Directrix $= y = -\frac{1}{4}$

$y = f(x)$
for this parabola

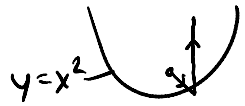


We know for every point on the parabola, including vertex, distance from focus to parabola = distance from parabola to directrix.

$$e = 1 = \frac{|PF|}{|PL|} = \frac{\sqrt{x^2 + (y-\frac{1}{4})^2}}{y + \frac{1}{4}} \quad \text{pythagorean theorem}$$

$$y + \tfrac{1}{4} = \sqrt{x^2 + (y - \tfrac{1}{4})^2}$$
$$y^2 + \tfrac{1}{2}y + \tfrac{1}{16} = x^2 + y^2 - \tfrac{1}{2}y + \tfrac{1}{16}$$
$$y = x^2$$
$$f(x) = x^2 \quad \text{and there's your parabola}$$

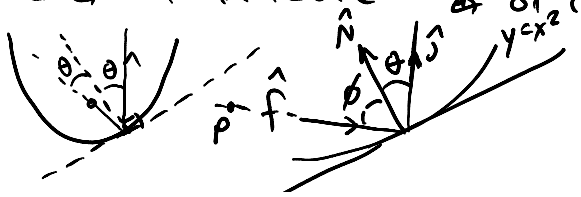
Special property of parabola



a light source will bounce it off parabola in such a way that its ∥ to y axis.

You can use that for headlights, receivers (put a dome there & then collect light/sound waves at focus, where you put your receiver.

The ∡ of incidence = ∡ of reflection.



$\hat{F} = \dfrac{\vec{PF}}{|\vec{PF}|}$

$\hat{N} \cdot \hat{T} = 0$ b/c ⊥

Prove that $\hat{F} \cdot \hat{N} = \hat{N} \cdot \hat{j}$ only true if $\theta = \phi$

b/c $|\hat{F}||\hat{N}|\cos\theta = |\hat{N}||\hat{j}|\cos\theta$ but unit vectors $=1$ so must be the $\cos\phi = \cos\theta$

X - FILES

$F \quad (0, \frac{1}{4})$

$P \quad (x, y=x^2) = (t, t^2)$

$\vec{PF} = \langle -t, \frac{1}{4} - t^2 \rangle$

Through simple subtraction

$|\vec{PF}| = \sqrt{t^2 + (\frac{1}{4} - t^2)^2}$

$$\frac{\vec{PF}}{|\vec{PF}|} = \frac{\langle -t, \frac{1}{4} - t^2 \rangle}{\sqrt{t^4 + \frac{1}{2}t^2 + \frac{1}{16}}} = \frac{\langle -t, \frac{1}{4} - t^2 \rangle}{\sqrt{(t^2 + \frac{1}{4})^2}}$$

$$= \frac{\langle -t, \frac{1}{4} - t^2 \rangle}{t^2 + \frac{1}{4}}$$

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\frac{d}{dt}\langle t, t^2 \rangle}{|\frac{d}{dt}\langle t, t^2 \rangle|} = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}}$$

$\hat{N} = \dfrac{\langle -2t, 1 \rangle}{\sqrt{1+4t^2}}$ dot product between $\hat{T}$ & $\hat{N}$ is 0, so this is right. reverse x & y coordinates & flip a sign to get the ⊥.

The flipping of the sign is the hard part b/c it needs to point in right direction.

Check ~~that~~ $\hat{F} \cdot \hat{N} = \dfrac{2t^2 + \frac{1}{4} - t^2}{(t^2 + \frac{1}{4})\sqrt{1+4t^2}} \stackrel{?}{=} \dfrac{1}{\sqrt{1+4t^2}}$

$$\hat{N} \cdot \hat{j} = \frac{1}{\sqrt{1+4t^2}}$$

Fns of several variables

Thursday, November 02, 2006
10:04 AM

Functions of several variables
Surfaces, continuity, Partial derivatives

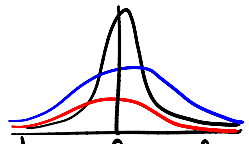
A function associates a unique number to a set of variables in a given domain.

$$z = f(x_1, x_2, \dots x_n) \; ; \; (x_1, x_2, \dots x_n) \in D$$

<u>Example</u>

$$T(x, y, z, t) = \frac{1}{(4\pi k t)^{3/2}} e^{-(x^2+y^2+z^2)/4kt}$$

Domain: $t > 0$, x, y, z $\mathbb{R}$



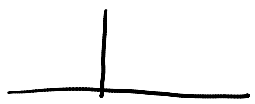
heat flow function.

T is temperature as a function of time t.

Δ Function is a function that is 0 everywhere but one point, where it's infinite.

If you integrate a Δ function, you get 1.

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$



$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$$

T = dependent variable
x, y, z, t = independent variables.

A surface is a 2D object in 3D space.
hypersurface = n-1 D object in n D space.

Explicit function of (xy) defines the surface.

$$z = f(x, y)$$

e.g. $z = \sqrt{x^2+y^2}$

An implicit def of a surface can be written in the form
$F(x, y, z) = 0$

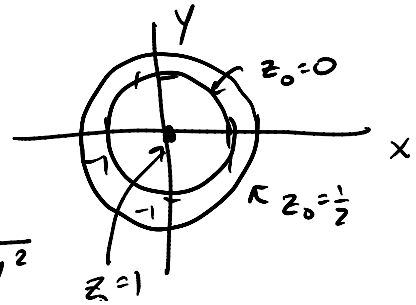
ex $x^2+y^2+z^2-1=0$ sphere of radius 1

$z = \sqrt{1-x^2-y^2}$
$z = -\sqrt{1-x^2-y^2}$ explicitly

explicit representation of a surface, then z is unique.
implicit representation $\Rightarrow$ possibly not unique

<u>Level curves</u> are curves that are formed by the intersection of a surface with planes of constant z i.e. ($z=c$)

<u>Contour Plots</u>



ex. $z = \sqrt{1-x^2-y^2}$
with
$z = z_0$
$\Rightarrow z_0 = \sqrt{1-x^2-y^2}$
$0 \le z \le 1$

We're only considering pos. values, so it's a hemisphere.

$z_0^2 = 1-x^2-y^2$
$x^2+y^2 = 1-z_0^2$
$0 \le z_0 \le 1$ Level curves will be circles, determined by $1-z_0^2$.

$z_0 = 0$
$0 = \sqrt{1-x^2-y^2}$
$x^2+y^2 = 1$ circle of radius 1.

$z_0 = \frac{1}{2}$
$\frac{1}{2} = \sqrt{1-x^2-y^2}$
$\Rightarrow x^2+y^2 = 1-\frac{1}{4} = \frac{3}{4}$

$z_0 = 1$
$1 = \sqrt{1-x^2-y^2}$
$x^2+y^2 = 0$ you get a point.

Level curves - other examples
Topographic Maps level curves of elevation
Isopressure curves weather maps. Pressure isobars

<u>Surfaces of revolution</u>
Height only depends on radius

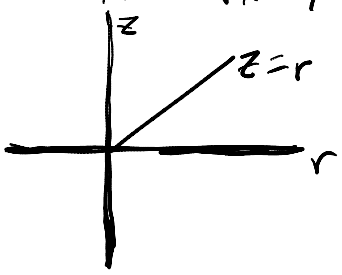
might only depends on radius

polar coordinates: $z = f(r) = f(\sqrt{x^2+y^2})$ rotating about the z axis.

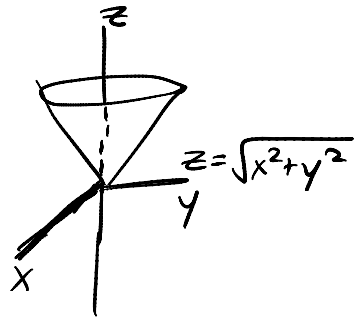
$x = r\cos\theta$
$y = r\sin\theta$

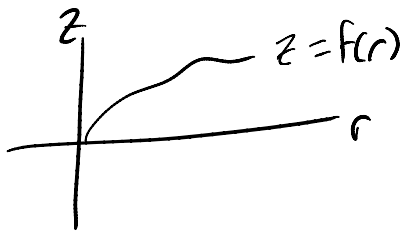
plot $z = f(r)$ or imagine rotating that about z axis.

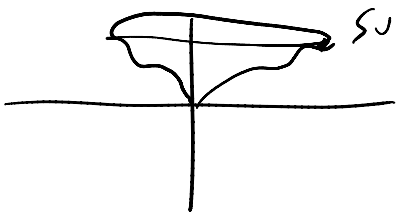
ex $z = \sqrt{x^2+y^2} = r$



rotate about z axis







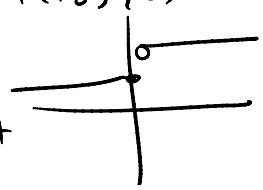
Surface of revolution

2D object, so it only defines the surface.

<u>Continuity</u>

No jumps (intuitively)

Mathematically, $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ } for all curves x_0, y_0



limits from both directions must also be the same.

ex. of function that fails to be continuous at some point.

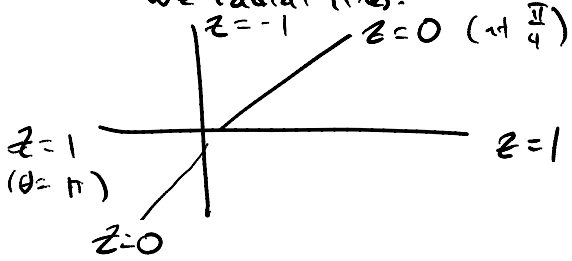
$z = \dfrac{x^2-y^2}{x^2+y^2}$ $(0,0)$

} possible problem at $(x_0,y_0) = (0,0)$

$x = r\cos\theta$
$y = r\sin\theta$
$z = r^2(\cos^2\theta - \sin^2\theta) = \cos 2\theta = \cos 2\theta$

$$z = \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = \frac{\cos 2\theta}{1} = \cos 2\theta$$

radial lines approaching the origin. Level curves of that function are radial lines.



if you approach it from diff angles, it has a diff value, so not continuous.

Partial Derivatives, Higher Derivatives

Friday, November 03, 2006
10:02 AM

A function of 2 variables.

<u>Partial Derivatives</u>

Given $z = f(x,y)$, the <u>partial</u> <u>derivative</u> of z with respect to x is defined as

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

It can be thought of as the derivative holding y constant

similarly $\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$

<u>ex.</u> $f = x^2 + 2y^2 + 2xy$

$\frac{\partial f}{\partial x} = 2x + 2y$ you treat y like it's a constant.

$\frac{\partial f}{\partial y} = 4y + 2x$ you treat x like it's a constant.

All the usual rules of differentiation apply:

Product Rule $\frac{\partial}{\partial x}\left(f(x,y)\, g(x,y)\right) = \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}$

$y = f(x)$

$\frac{dy}{dx} = f'(x)$ but you can't put a prime there anymore b/c we don't know what we're deriving w/ respect to.

Notation $z = f(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y$$

<u>Higher derivatives</u>

$\frac{\partial^2 f}{\partial x^2}$ 2nd partial derivative w.r.t x (w/ respect to) $= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

$$= f_{xy} \ ?$$

For a lot of functions, the order doesn't matter.

If f_{xy} & f_{yx} are continuous at $(x,y) = (a,b)$ then $f_{xy}(a,b) = f_{yx}(a,b)$

$f(x,y) = e^{xy^2}$ $\quad f_x, f_y, f_{xy}, f_{yx}$

$f_x = \frac{\partial f}{\partial x} = y^2 e^{xy^2}$ b/c y^2 is just a constant.

$f_y = \frac{\partial f}{\partial y} = 2xy\, e^{xy^2}$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y^2 e^{xy^2}\right) = 2y e^{xy^2} + y^2 e^{xy^2}(2xy)$$

$$= e^{xy^2}(2y + 2xy^3)$$

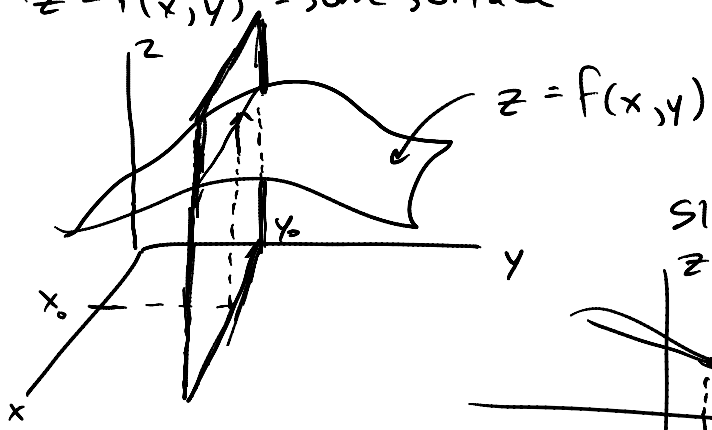
$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2xy e^{xy^2}\right) = 2y e^{xy^2} + 2xy\, e^{xy^2} y^2$$

$$= e^{xy^2}(2y + 2xy^3)$$

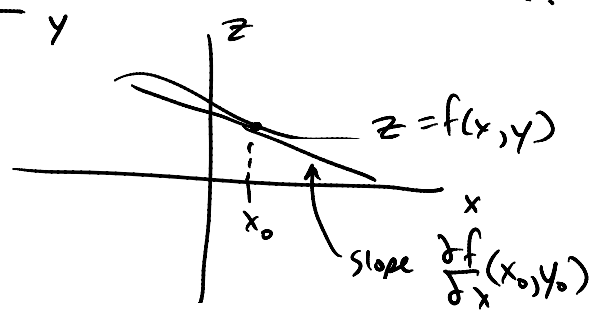
So they're continuous b/c you get the same value

Interpret $\frac{\partial f}{\partial x}(x_0, y_0)$ & $\frac{\partial f}{\partial y}(x_0, y_0)$

$z = f(x,y)$ = some surface



Slice the surface.

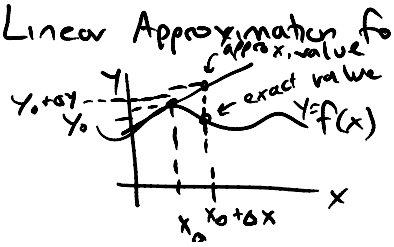


Differentials, Tangent Plane, Linear Approx

Monday, November 06, 2006
10:03 AM

differentials
tangent plane
linear approximation

Linear Approximation for a function of a single variable



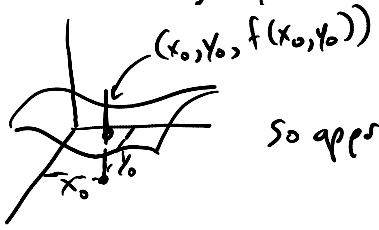
Slope of tangent line $= \frac{df}{dx}(x_0)$

$$\frac{\Delta y}{\Delta x} \approx \frac{df}{dx}(x_0)$$

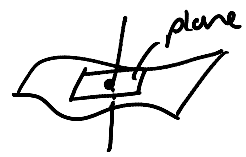
$$y - y_0 = \frac{df}{dx}(x_0)(x - x_0)$$

$$y \approx f(x)$$

equation for tangent plane to surface $z = f(x,y)$ at $(x,y) = (x_0, y_0)$



So approximate a plane



Take 2 vectors that lie in the plane, $\vec{AB}$ & $\vec{BC}$, & find normal vector $\vec{N} \perp$ 2 vectors in the plane. $\vec{N} = \langle a, b, c \rangle$

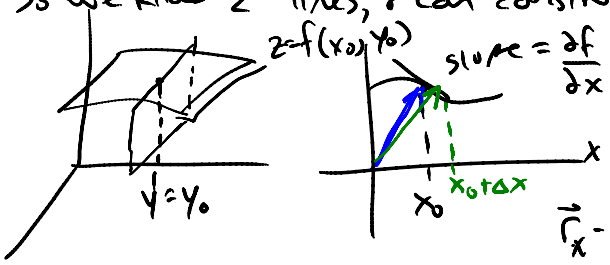
$\langle x_0, y_0, z_0 \rangle$ is in the plane $z_0 = f(x_0, y_0)$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = d$, where $d = -(-ax_0 - by_0 - cz_0)$

$= ax_0 + by_0 + cz_0$

Partial derivative tells you rate of change in a particular direction.
So we know 2 lines, & can construct 2 vectors in this way.



$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{r}_x = \langle x_0 + \Delta x, y_0, z_0 + \frac{\partial f}{\partial x}\Delta x \rangle$

$\vec{r}_x - \vec{r}_0 = \vec{u} = \langle \Delta x, 0, \frac{\partial f}{\partial x}\Delta x \rangle$

$\vec{u} = \langle \Delta x, 0, \frac{\partial f}{\partial x}\Delta x \rangle$ holding y constant

$\vec{v} = \langle 0, \Delta y, \frac{\partial f}{\partial y}\Delta y \rangle$

$\vec{v} = \langle 0, \Delta y, \frac{\partial f}{\partial y}\Delta y \rangle$

$$\vec{n} = |\vec{u} \times \vec{v}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x}\Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y}\Delta y \end{vmatrix} = \hat{i}\left(0 - \frac{\partial f}{\partial x}\Delta x \Delta y\right) - \hat{j}\left(0 - \frac{\partial f}{\partial x}\Delta x \Delta y\right) + \hat{k}\left(0 - \frac{\partial f}{\partial y}\Delta x \Delta y\right)$$

$$= -\hat{i}\frac{\partial f}{\partial x}\Delta x \Delta y + \hat{j}\frac{\partial f}{\partial x}\Delta x \Delta y - \hat{k}\frac{\partial f}{\partial x}\Delta x \Delta y$$

$$\vec{N} = \Delta x \Delta y \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

$$\langle a, b, c \rangle$$

Tangent Plane

$$\boxed{-\frac{\partial f}{\partial x}(x - x_0) - \frac{\partial f}{\partial y}(y - y_0) + (z - z_0) = 0}$$

Linear approximation formula: if we let $z - z_0 = \Delta z$

$y - y_0 = \Delta y$

$x - x_0 = \Delta x$

$$\boxed{\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}$$

$\Delta y = \frac{df}{dx}\Delta x$ for linear approx for single variable.

Let's estimate something w/o calculator:

ex Linear Approx to estimate $\sqrt{3.02^2 + 3.97^2}$

$z = f(x, y)$

choose and (x_0, y_0)

$x = 3.02$

$y = 3.97$

$z = \sqrt{x^2 + y^2}$

$(x_0 = 3, y_0 = 4)$

$z_0 = \sqrt{3^2 + 4^2} = 5$

$$f(x,y) = z \approx z_0 + \frac{\partial f}{\partial x}(x_0, y_0)\underbrace{(x - x_0)}_{.02} + \frac{\partial f}{\partial y}(x_0, y_0)\underbrace{(y - y_0)}_{-.03}$$

$f = \sqrt{x^2 + y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial f}{\partial x}(3,4) = \frac{3}{\sqrt{3^2 + 4^2}} = \frac{3}{5}$$

$$\frac{\partial f}{\partial y}(3,4) = \frac{4}{\sqrt{3^2 + 4^2}} = \frac{4}{5}$$

$z_0 = 5$

$$f(x,y) = z \approx 5 + \frac{3}{5}(.02) + \frac{4}{5}(-.03)$$

$$\approx 5 + \frac{.12 - .24}{10}$$

$$\approx 5 + \frac{-.12}{10} = 5 - .012 = 4.988$$

exactly, it's 4.9881. so it agrees to 4 decimal places.

<u>differential notation</u>

$y = f(x)$

Linear approximation $\Delta y \approx \frac{df}{dx}\Delta x$

Let $\Delta x \to 0$, then we write $dy = \frac{df}{dx}\Delta x$

$$dy = \frac{dy}{dx}dx$$

$$du = \frac{du}{dx}dx$$

when $z = f(x,y)$

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$\Delta x \to 0$ & $\Delta y \to 0$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Application of differential notation

$z = f(x,y)$ & $y = y(x)$, find $\frac{dz}{dx}$

recall $dz = \underset{\frac{\partial f}{\partial x}}{f_x}dx + \underset{\frac{\partial f}{\partial y}}{f_y}dy$ & $dy = \frac{dy}{dx}dx$

$$dz = f_x dx + f_y \frac{dy}{dx}dx$$

$$\boxed{\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}}$$ Implicit differentiation.

We can use this to define an operator

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}$$ we just took out the f

For tomorrow

if $f(x,y) = 0$ defines a curve implicitly, Compute $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in terms of partial derivatives

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Show $\frac{d^2y}{dx^2} = \underline{2 f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}}$

Show $\frac{d^2y}{dx^2} = \frac{2 f_x f_y t_{xy} - t_x^2 t_{yy} - t_y^2 t_{xx}}{f_y^2}$

Chain Rule

Tuesday, November 07, 2006
10:03 AM

Suppose $f(x,y)=0$ ← implicitly defines a curve in 2D
$(y=y(x))$

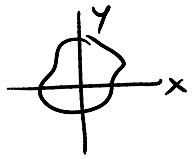
Compute $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in terms of partial derivatives of f.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad \text{From yesterday}$$

$$+\ dy = \frac{dy}{dx}dx$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dx}dx$$

df is 0 b/c $f(x,y)=0$
$z=0$



no z movement

$$0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dx}dx$$

$$0 = \underbrace{dx}_{0 \text{ or } 0}\underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)}_{0}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-f_x}{f_y}$$

$$f = x^3 + y^3 + 1 = 0$$

$$\frac{dy}{dx}\ ?$$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-3x^2}{3y^2} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

so we've always had partial derivatives. we just didn't know it.

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{dy}{dx}\frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)f$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{f_x}{f_y}\right)$$

$$= -\underbrace{\left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)}_{\frac{d}{dx}}\left(\frac{f_x}{f_y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{f_x}{f_y}\right)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

$$= \frac{f_{xx}f_y - f_x f_{yx}}{f_y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{f_x}{f_y}\right) = \frac{f_{xy}f_y - f_x f_{xy}}{f_y^2}$$

$$\frac{\partial^2 y}{\partial x^2} = -\left[\frac{f_{xx}f_y - f_x f_{yx}}{f_y^2} - \frac{f_x}{f_y}\left(\frac{f_{xy}f_y - f_x f_{xy}}{f_y^2}\right)\right]$$

$$\frac{\partial y}{\partial x^2} = \left[\frac{f_{xx}f_y - f_x f_{yx}}{f_y^2} - \frac{f_x}{f_y}\left(\frac{f_{xy}f_y - f_x f_{xy}}{f_y^2}\right)\right]$$

$$\frac{\partial^2 y}{\partial x^2} = -\left[\left(\frac{f_{xx}f_y - f_x f_{yx}}{f_y^2}\right) + \frac{f_x}{f_y}\left(\frac{f_{xy}f_y - f_x f_{xy}}{f_y^2}\right)\right]$$

f_y

end result. $\rightarrow$ $$\frac{\partial^2 y}{\partial x^2} = \frac{f_{xx}f_y f_{yx} - f_{xx}f_y^2 + f_x f_y f_{xy} - f_x^2 f_{yy}}{f_y^3}$$

$$\frac{\partial^2 y}{\partial x^2} = -\left[\underbrace{\frac{f_{xx}f_y - f_x f_{yx}}{f_y^2}}_{\frac{\partial}{\partial x}\left(\frac{f_x}{f_y}\right)} + \underbrace{\left(-\frac{f_x}{f_y}\right)}_{\frac{dy}{dx}}\underbrace{\left(\frac{f_{xy}f_y - f_x f_{yy}}{f_y^2}\right)}_{\frac{\partial}{\partial y}\left(\frac{f_x}{f_y}\right)}\right]$$

$$\frac{-f_{xx}f_y^2 + f_x f_y f_{yx} + f_x f_y f_{xy} - f_x^2 f_{yy}}{f_x^3}$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$\frac{d^2y}{dx^2} = \frac{2f_x f_y f_{yx} - f_{xx}f_y^2 - f_x^2 f_{xy}}{f_y^3}$$

$f = x^3 + y^3 + 1 = 0$
$f_x = 3x^2$
$f_y = 3y^2$
$f_{xx} = 6x$
$f_{yy} = 6y$
$f_{yx} = f_{xy} = 0$

applies if f only depends on x. $\frac{df}{dx}$
$\frac{\partial f}{\partial x}$ treats y as an independent variable

$$\frac{d^2y}{dx^2} = \frac{-6x(3y^2)^2 - (3x^2)^2(6y)}{(3y^2)^3} = \frac{-54xy^4 - 54x^4y}{27y^6}$$

$$= \frac{-54xy^3 - 54x^4}{27y^5}$$

$$= \frac{-2x}{y^2} - \frac{2x^4}{y^5}$$

Chain Rule

Calculus of a single variable

if $f = f(x)$ and $x = x(t)$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Interpret as a related rates problem $\rightarrow$ time rate of change of f is equal to rate of change of f w.r.t. x times time rate of change of x

3D: $f = f(x, y, z)$
$x = x(t)$
$y = y(t)$
$z = z(t)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$dx = \frac{dx}{dt}dt \qquad dy = \frac{dy}{dt}dt \qquad dz = \frac{dz}{dt}dt$$

$$df = \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt + \frac{\partial f}{\partial z}\frac{dz}{dt}dt$$

$$df = \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt + \frac{\partial f}{\partial z}\frac{dz}{dt}dt$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$ <u>Chain rule</u>

$$\vec{r} = \langle x, y, z \rangle$$
$$\vec{v} = \frac{df}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$\nabla f = \text{grad}(f)$

Gradient of f = vector

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$f = xyz$, $\nabla f = \langle yz, xz, xy \rangle$

$\frac{df}{dt}$ is just a dot product between the velocity & the gradient.

$$\frac{df}{dt} = \nabla f \cdot \vec{v}$$

<u>Ex</u>

Consider the path $\vec{r} = \langle x, y, z \rangle = \langle t, t^2, t^3 \rangle$. How fast does the pt. move away from the origin?

Distance from origin in terms of x, y, z $= \sqrt{x^2+y^2+z^2}$

$$\frac{dD}{dt} = \nabla D \cdot \vec{v}$$

$$\nabla D = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$
$$= \left\langle \frac{2x}{2\sqrt{x^2+y^2+z^2}}, \frac{2y}{2\sqrt{x^2+y^2+z^2}}, \frac{2z}{2\sqrt{x^2+y^2+z^2}} \right\rangle$$
$$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$\frac{dD}{dt} = \nabla D \cdot \vec{v} = \left\langle \frac{x}{D}, \frac{y}{D}, \frac{z}{D} \right\rangle \cdot \langle 1, 2t, 3t^2 \rangle$$
$$= \frac{x}{D} + \frac{2ty}{D} + \frac{3t^2 z}{D} = \frac{t + 2t^2 + 3t^3}{\sqrt{t^2 + t^4 + t^6}}$$

Directional Derivative

Wednesday, November 08, 2006
9:55 AM

Directional Derivative

Gradient of a function $f(x,y,z)$ is a vector field

$$\nabla f = \text{grad}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Properties of ∇f IMPORTANT

1) The gradient points in the direction of maximum increase in f. } defer proof
2) The gradient is $\perp$ to the level surfaces of $f(x,y,z)$.

Level surface, $C = f(x,y,z)$

curve of a particle in level surface. $r(t) = \langle x(t), y(t), z(t) \rangle$

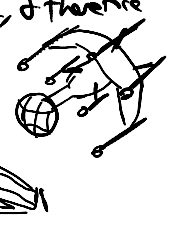
$$\frac{df}{dt} = \frac{d}{dt}\Big[f(x(t)), y(t), z(t))\Big] = \frac{d}{dt}C = 0$$

And so it's $=0$, b/c function doesn't change.

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$= \nabla f \cdot \vec{v}$ we know the dot product $=0$, and so this tells us that $\nabla f \perp \vec{v}$.

$\vec{v}$ is in the direction that is tangent to the surface, & therefore we know that ∇f is $\perp$ to level curves/surface



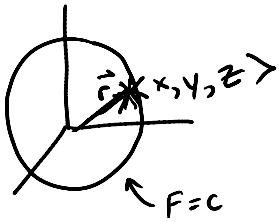
Consider the following function: $f = x^2 + y^2 + z^2$.

Spherical shells are the level curves.

1) Find a normal vector to the level surfaces $f = c$
2) Find the equation for the tangent plane to the level surface at (x_0, y_0, z_0)

1) $\vec{N} = \nabla f$ b/c that gives you a vector $\perp$ to level surfaces.

$\vec{N} = \langle 2x, 2y, 2z \rangle = 2\langle x, y, z \rangle$



2) Equation: (for a plane)

$$\vec{N}\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$$

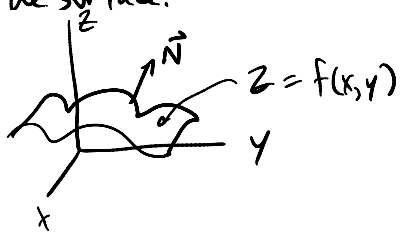
$$\vec{N} = 2\langle x_0, y_0, z_0\rangle.$$

$$2x_0(x-x_0) + 2y_0(y-y_0) + 2z_0(z-z_0) = 0$$

$$2xx_0 + 2yy_0 + 2zz_0 = 2x_0^2 + 2y_0^2 + 2z_0^2$$

$$\boxed{xx_0 + yy_0 + zz_0 = x_0^2 + y_0^2 + z_0^2}$$

<u>Ex</u>. Suppose $z = f(x,y)$ defines a surface. Find a normal vector to the surface.



Let $F(x,y,z) = z - f(x,y)$
$= 0$

There's a little trick.

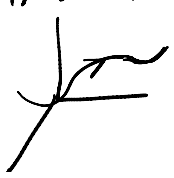
$$\boxed{\nabla F = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle}$$

k component is pos, so it points in the upward direction.

$\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle$ is your $\perp$ normal vector.

<u>Gradient & the directional derivative</u>

Recall if $F = F(x,y,z)$ & $x = x(t)$, $y = y(t)$, $z = z(t)$



$$\Rightarrow \boxed{\frac{dF}{dt} = \nabla F\cdot\vec{v}}$$

Chain rule applied to functions of several variables.

depends on parameterization w.r.t t.
Not inherent of the curve.

To determine an inherent property of the curve, calculate rate of change w.r.t. arclength.

$$\frac{dF}{ds} = \frac{dF}{dt}\underbrace{\frac{dt}{ds}}_{\frac{1}{|\vec{v}|}} = (\nabla F\cdot\vec{v})\frac{1}{|\vec{v}|} = \nabla F\cdot\frac{\vec{v}}{|\vec{v}|} = \boxed{\nabla F\,\hat{T}}$$

This is an example of a directional derivative.

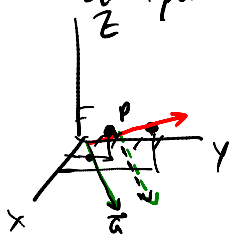
Directional derivatives tell us that all we have to

do to calculate the rate of change of $F(x,y,z)$ in the direction of $\hat{u}$ is the following "directional derivative": $\nabla F \cdot \hat{u}$

unit vector

You can figure out your function in any direction.

<u>Ex.</u> Find the rate of change of $F = x^2+y^2+z^2$ in the direction $\vec{a}=\langle 1,1,0\rangle$ at a particular point, $(1,2,1)$.



$$\nabla F \cdot \hat{u}$$

$$\nabla F = \langle 2x, 2y, 2z\rangle$$

$$\nabla F\Big|_{(1,2,1)} = \langle 2, 4, 2\rangle$$

$$\frac{\vec{a}}{|\vec{a}|} = \frac{\langle 1,1,0\rangle}{\sqrt{2}}$$

$$\langle 2,4,2\rangle \cdot \frac{\langle 1,1,0\rangle}{\sqrt{2}}$$

$$= \frac{2+4+0}{\sqrt{2}} = \frac{6}{\sqrt{2}} = \boxed{3\sqrt{2}}$$

Directional Derivatives 2

Friday, November 10, 2006
10:09 AM

Use Directional Derivative to show that the gradient points in the direction of maximum increase in $F(x,y,z)$

$\frac{dF}{ds}$ = directional derivative in direction of $\hat{T}$. $\nabla F \cdot \hat{T}$ rate of change w/ respect to arc-length in that direction.

$\nabla F \cdot \hat{T} = |\nabla F||\hat{T}|\cos\theta$

$|\hat{T}| = 1$ b/c it's a unit vector

$|\nabla F|\cos\theta$

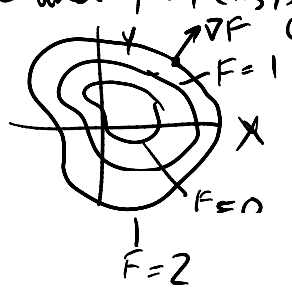
Largest it could be is when $\cos\theta = 1$, $\theta = 0$

$\Rightarrow \frac{dF}{ds} = |\nabla F|$, so $\hat{T} \parallel \nabla F$

Magnitude of gradient tells you rate of change of function in the direction where it changes most rapidly.

∇F = Max rate of change of F

picture when $F = F(x,y)$



Gradient is $\perp$ to level curves.

Gradient is a 2D vector here, but when level curves are surfaces in 3D, Gradient is 3D vector.

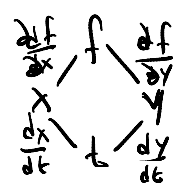
Dot product has to be 0, so $\perp$.
b/c $\theta = 0$ (max rate of change)

Chain Rule

$f(x,y)$

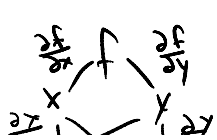
$x = x(t)$, $y = y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \vec{v}$$



now consider a change of variables from (x,y) to (r,s)
Cartesian (x,y) to Polar (r,θ)

$f(x,y) = f(x(r,s), y(r,s))$



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$] rate of change of f w.r.t. r, s held

$\frac{\partial x}{\partial r}$ $\frac{\partial y}{\partial s}$ r $\frac{\partial y}{\partial r}\frac{\partial x}{\partial s}$ s

constant

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

ex. $f = \ln(x) + \ln(y)$ $\qquad r>0$

$x = re^{s}$ & $y = re^{-s}$ $\qquad s = \mathbb{R}$

$\frac{\partial f}{\partial r}$ & $\frac{\partial f}{\partial s}$?

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \qquad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial x} = \frac{1}{x} = \frac{1}{re^{s}} \qquad \frac{\partial x}{\partial r} = e^{s} \qquad \frac{\partial y}{\partial r} = e^{-s}$$

$$\frac{\partial f}{\partial y} = \frac{1}{y} = \frac{1}{re^{-s}} \qquad \frac{\partial x}{\partial s} = re^{s} \qquad \frac{\partial y}{\partial s} = -re^{-s}$$

$$\frac{\partial f}{\partial r} = \left(\frac{1}{re^{s}}\right)\left(e^{s}\right) + \left(\frac{1}{re^{-s}}\right)\left(e^{-s}\right) = \frac{2}{r}$$

$$\frac{\partial f}{\partial s} = \left(\frac{1}{re^{s}}\right)\left(re^{s}\right) + \left(\frac{1}{re^{-s}}\right)\left(-re^{-s}\right) = 0$$

This function is independent of the variable s.

$$f = \ln(x) + \ln(y)$$
$$f = \ln(re^{s}) + \ln(re^{-s})$$
$$f = \ln(re^{s}re^{-s}) = \ln(r^2)$$
$$= 2\ln r$$

s doesn't appear at all, so it is independent of s.

$$\frac{\partial f}{\partial r} = \frac{2}{r} \checkmark$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$ Partial differential equation. "Laplace's Eqn."

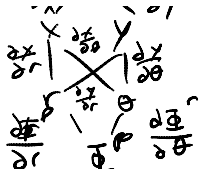
What this eqn is if written in polar coordinats?

$$\Phi^c = \Phi(x,y) = \Phi(r\cos\theta, r\sin\theta) = \Phi^c(r,\theta)$$

What is $\frac{\partial \Phi}{\partial x}$ & $\frac{\partial \Phi}{\partial y}$ in terms of r & θ where $x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial r}{\partial x}$ Φ^c $\frac{\partial \Phi^c}{\partial y}$

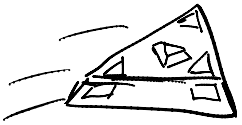
$$\frac{\partial \Phi}{\partial \quad} = \frac{\partial \Phi}{\partial \quad}\frac{\partial x}{\partial \quad} + \frac{\partial \Phi}{\partial \quad}\frac{\partial y}{\partial \quad}$$ Solve for



$$\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial \Phi}{\partial \theta} = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial \theta}$$

Solve for $\frac{\partial \Phi}{\partial x}$ & $\frac{\partial \Phi}{\partial y}$



OR

$$\frac{\partial \Phi^c}{\partial x} = \frac{\partial \Phi^p}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi^p}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial \Phi^c}{\partial y} = \frac{\partial \Phi^p}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Phi^p}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$x = r\cos\theta \quad y = r\sin\theta \qquad r^2 = x^2 + y^2, \ \tan\theta = \frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}\cos^2\theta = -\frac{\sin\theta}{r}$$

$$\frac{\partial}{\partial x}(r^2 - x^2 + y^2)$$

$$2r\frac{\partial r}{\partial x} - 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{1}{x}\cos^2\theta = \frac{\cos\theta}{r}$$

$$\boxed{\frac{\partial \Phi}{\partial x} = \cos\theta\frac{\partial \Phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \Phi}{\partial \theta}}$$

$$\frac{\partial}{\partial x}\left(\tan\theta = \frac{y}{x}\right)$$

$$\sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}\cos^2\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{x}\cos^2\theta$$

Chain Rule

Monday, November 13, 2006
10:02 AM

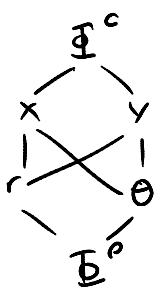
Chain Rule

$$\Phi(x,y) = \Phi(r\cos\theta, r\sin\theta) = \Phi^p(r,\theta)$$

$x = r\cos\theta$
$y = r\sin\theta$

1) Write $\frac{\partial \Phi^c}{\partial x}$ & $\frac{\partial \Phi^c}{\partial y}$ in terms of $\frac{\partial \Phi^p}{\partial r}$ & $\frac{\partial \Phi^p}{\partial \theta}$

2) Write $\frac{\partial^2 \Phi}{\partial x \partial y}$ in terms of Φ_{rr} $\Phi_{\theta\theta}$



$$\Phi^c(x,y) = \Phi(x(r,\theta), y(r,\theta))$$
$$\Phi^p(r,\theta) = \Phi(r(x,y), \theta(x,y))$$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \begin{array}{l} r^2 = x^2+y^2 \ ; \\ \tan\theta = \frac{y}{x} \end{array}$$

$$\frac{\partial}{\partial x}(r^2 = x^2+y^2) \Rightarrow 2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial}{\partial y}(r^2 = x^2+y^2) \Rightarrow 2r\frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial x} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial}{\partial x}\left(\tan\theta = \frac{y}{x}\right) \Rightarrow \sec^2\theta\frac{\partial \theta}{\partial y} = -\frac{y}{x^2}$$

$$\frac{\partial}{\partial y}\left(\tan\theta = \frac{y}{x}\right) \Rightarrow \sec^2\theta\frac{\partial \theta}{\partial y} = \frac{1}{x}$$

$$\frac{\partial \theta}{\partial x} = -\cos^2\theta \cdot \frac{r\sin\theta}{(r\cos\theta)^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos^2\theta}{r\cos\theta} = \frac{\cos\theta}{r}$$

$$\Phi_x = \frac{\partial \Phi}{\partial r}\cos\theta - \frac{\partial \Phi}{\partial \theta}\frac{\sin\theta}{r} = \Phi_r\cos\theta - \Phi_\theta\frac{\sin\theta}{r}$$

<u>Important observation:</u>

$$\frac{\partial r}{\partial x} \neq \frac{1}{\left(\frac{\partial x}{\partial r}\right)}$$

$$\frac{\partial r}{\partial x} = \cos\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta$$

If you took $\frac{1}{\frac{\partial x}{\partial r}}$, you'd get $\sec\theta$.

So $\frac{\partial r}{\partial x} \neq \left(\frac{1}{\left(\frac{\partial x}{\partial r}\right)}\right)$

$$\Phi_x = \frac{\partial \Phi}{\partial r}\cos\theta - \frac{\partial \Phi}{\partial \theta}\frac{\sin\theta}{r} = \Phi_r \cos\theta - \Phi_\theta \frac{\sin\theta}{r}$$

$$\Phi_y = \frac{\partial \Phi}{\partial r}\sin\theta + \frac{\partial \Phi}{\partial \theta}\frac{\cos\theta}{r} = \Phi_r \sin\theta + \Phi_\theta \frac{\cos\theta}{r}$$

2) $\frac{\partial r}{\partial x} = \cos\theta \quad \frac{\partial \theta}{\partial x} = \frac{\sin\theta}{r} \qquad \frac{\partial r}{\partial y} = \sin\theta \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial}{\partial y}\Big(\underbrace{\cos\theta\, \Phi_r - \frac{\sin\theta}{r}\Phi_\theta}_{G(r,\theta)}\Big)$$

$$= \frac{\partial}{\partial r}\left(\cos\theta\, \Phi_r - \frac{\sin\theta}{r}\Phi_\theta\right)\frac{\partial r}{\partial y}$$

$$+ \frac{\partial}{\partial \theta}\left(\cos\theta\, \Phi_r - \frac{\sin\theta}{r}\Phi_\theta\right)\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial}{\partial r}\left(\cos\theta\, \Phi_r - \frac{\sin\theta}{r}\Phi_\theta\right)\sin\theta$$
$$+ \frac{\partial}{\partial \theta}\left(\cos\theta\, \Phi_r - \frac{\sin\theta}{r}\Phi_\theta\right)\frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial r}\Big(\underbrace{\cos\theta}_{\text{constant}}\, \underbrace{\Phi_r}_{\Phi_{rr}} - \underbrace{\frac{\sin\theta}{r}}_{\frac{-\sin\theta}{r^2}}\underbrace{\Phi_\theta}_{-\frac{\sin\theta}{r}\Phi_{\theta r}}\Big) = \cos\theta\, \Phi_{rr} + \frac{\sin\theta}{r^2} - \frac{\sin\theta}{r}\Phi_{\theta r}$$

Product Rule

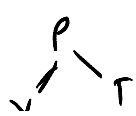
$$\frac{\partial}{\partial \theta}\left(\cos\theta\, \Phi_r - \frac{\sin\theta}{r}\Phi_\theta\right) = -\sin\theta\, \Phi_r + \cos\theta\, \Phi_{r\theta} - \frac{\cos\theta}{r}\Phi_\theta - \frac{\sin\theta}{r}(\Phi_{\theta\theta})$$

Check by computing $\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial y}\right)$

Implied (partial) differentiation

<u>Ex</u> Ideal Gas Law $\quad PV = nRT$

Find the change in pressure w.r.t. V, at constant T.



$$\left(\frac{\partial P}{\partial V}\right)_T$$

$$\frac{\partial}{\partial V}(PV = nRT)_T$$

$P = P(V,T)$

$$\frac{\partial P}{\partial V}V + P\frac{\partial V}{\partial V} = 0 \quad (\text{b/c constant}) \qquad \frac{\partial P}{\partial V} = \frac{-P}{V}$$

$$\left(\frac{\partial P}{\partial V}\right) = -\frac{P}{V}$$

$$\left(\frac{\partial \rho}{\partial V}\right)_T = -\frac{\rho}{V}$$

Suppose $x = f(r,s)$
$y = g(r,s)$

old variables

new variables

e.g. $x = r\cos\theta = f(r,\theta)$
$y = r\sin\theta = g(r,\theta)$

use old variables to compute $\left(\frac{\partial r}{\partial x}\right)_y$

so we need to know how x & r relate.

differentiate implicitly.

$$\frac{\partial}{\partial x}\Big|_{y\,\text{constant}} (x = f(r,s)) : \; 1 = \frac{\partial}{\partial x}\big(f(r(x,y), s(x,y))\big)$$

$$\frac{\partial}{\partial x}\Big|_{y\,\text{constant}} (y = g(r,s)) : \; 0 = \frac{\partial}{\partial x}\big(g(r(x,y), s(x,y))\big)$$

$$1 = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} \qquad 0 = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial x}$$

We want everything in terms of r & s.

So let's use both eqns to eliminate x & y dependence.

$$1 = f_r \frac{\partial r}{\partial x} + f_s \frac{\partial s}{\partial x}$$

$$0 = g_r \frac{\partial r}{\partial x} + g_s \frac{\partial s}{\partial x}$$

$$\frac{\partial s}{\partial x} = -\frac{g_r}{g_s}\frac{\partial r}{\partial x}$$

$$1 = f_r \frac{\partial r}{\partial x} + f_s\left(-\frac{g_r}{g_s}\frac{\partial r}{\partial x}\right)$$

$$1 = \frac{\partial r}{\partial x}\left(f_r - f_s\left(\frac{g_r}{g_s}\right)\right)$$

$$1 = \frac{\partial r}{\partial x}\left(\frac{f_r g_s - f_s g_r}{g_s}\right)$$

$$\frac{g_s}{f_r g_s - f_s g_r} = \frac{\partial r}{\partial x}$$

Check the case $\left.\begin{array}{l} x = r\cos\theta = f(r,\theta) \\ y = r\sin\theta = g(r,\theta) \end{array}\right\}$ $\frac{\partial r}{\partial x} = \cos\theta$ from before

$$\frac{\partial r}{\partial x} = \frac{g_\theta}{f_r g_\theta - f_\theta g_r}$$

$$\frac{\partial g}{\partial \theta} = r\cos\theta \qquad \frac{\partial f}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial f}{\partial r} = \cos\theta \qquad \frac{\partial g}{\partial r} = \sin\theta$$

$$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{(\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta)}$$

$$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta$$

Implicit Partial Differentiation

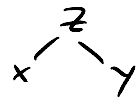
Tuesday, November 14, 2006
10:04 AM

Implicit Partial Differentiation

<u>Problem 1</u> $f(x,y,z)=0$ implicitly defines a surface in 3D space.

What is $\left(\frac{\partial z}{\partial x}\right)_y$? ⇐ partial z w.r.t. x, y held constant

other notation $\left.\frac{\partial z}{\partial x}\right|_y$



$$\frac{\partial}{\partial x}\left[f(\underline{x}, y, \underline{z}) = 0\right]$$

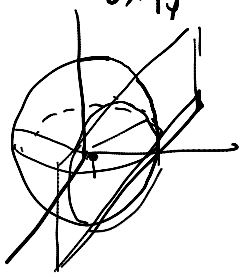
$z = z(x,y)$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

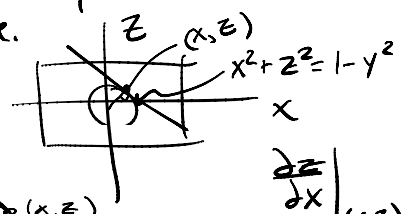
$$\frac{\partial z}{\partial x} = \frac{\left(-\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial z}\right)} = \frac{-f_x}{f_z}$$

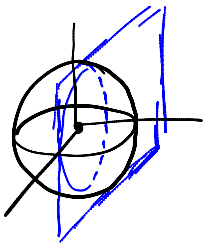
<u>ex.</u> $f = x^2 + y^2 + z^2 - 1 = 0$

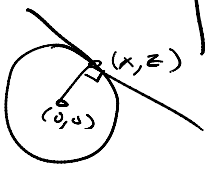
$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{f_x}{f_z} = -\frac{2x}{2z} = \frac{-x}{z}$$



When a plane slices the sphere, you get a circle.



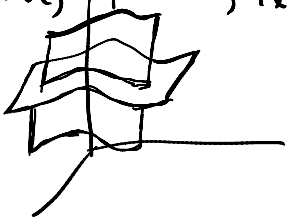




$\left.\frac{\partial z}{\partial x}\right|_{(x,z)} = -\frac{x}{z}$ b/c it's the opposite reciprocal of slope of line from $(0,0)$ to (x,z)

Take 2 surfaces that intersect at a curve & find tangent line.

<u>Problem 2</u> Suppose $z = f(x,y)$ & $z = g(x,y)$, & they intersect along a curve, $\vec{r} = \langle x, y(x), z(x) \rangle$



Find a tangent vector to the curve in terms of f_x, f_y, g_x, g_y

$\vec{T} = \frac{\partial \vec{r}}{\partial x} = \langle 1, \frac{dy}{dx}, \frac{dz}{dx} \rangle$ is tangent to the curve

$\frac{dy}{dx}$ & $\frac{dz}{dx}$ we know $z = f(x,y)$ & $z = g(x,y)$ on the curve.

$z = f(x, y(x))$ $\qquad$ $z = g(x, y(x))$

$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ $\qquad$ $\frac{dz}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx}$

$\frac{dz}{dx} = f_x + f_y \frac{dy}{dx}$ $\qquad$ 2 eqns, 2 unknowns

$\frac{dz}{dx} = g_x + g_y \frac{dy}{dx}$

$g_y \frac{dz}{dx} = g_y f_x + f_y g_y \frac{dy}{dx}$

$f_y \frac{dz}{dx} = f_y g_x + f_y g_y \frac{dy}{dx}$

$\frac{dz}{dx}(g_y - f_y) = g_y f_x - f_y g_x$

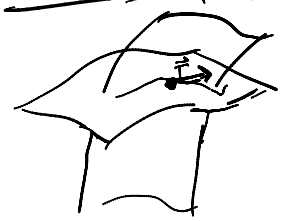
$\vec{T} = \langle 1, \quad , \frac{g_y f_x - f_y g_x}{g_y - f_y} \rangle$ $\qquad$ $\frac{g_y f_x - f_y g_x}{g_y - f_y} = g_x + g_y \frac{dy}{dx}$

$\vec{T} = \langle 1, \frac{f_x - g_y}{g_y - f_y}, \frac{g_y f_x - f_y g_x}{g_y - f_y} \rangle$ $\qquad$ $\frac{g_y f_x - f_y g_x}{g_y(g_y - f_y)} - \frac{g_x}{g_y} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{g_y}\left(\frac{g_y f_x - \cancel{f_y g_x} - g_x g_y + \cancel{g_x f_y}}{g_y - f_y}\right)$

$\frac{dy}{dx} = \frac{f_x - g_x}{g_y - f_y}$

Problem 2 Method 2



The tangent, $\vec{T}$ is $\perp$ to normal vector.

Let $\vec{N}_1$ be $\perp$ to $z = f(x,y)$ & $\vec{N}_2 \perp$ $z = g(x,y)$

$\vec{T} \perp \vec{N}_1$ & $\vec{T} \perp \vec{N}_2$

Cross product of $\vec{N}_1$ & $\vec{N}_2$ gives you a vector $\perp$ to both of them. ($\vec{T}$)

The gradient of the function, z, gives you $\vec{N}_1$ & $\vec{N}_2$.

$\vec{N}_1 = \nabla F_1$ $\qquad$ $F_1 = z - f(x,y)$

$\vec{N}_2 = \nabla F_2$ $\qquad$ $F_2 = z - g(x,y)$

$\vec{N}_1 \times \vec{N}_2 = \langle -f_x, -f_y, 1 \rangle \times \langle -g_x, -g_y, 1 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -f_x & -f_y & 1 \\ -g_x & -g_y & 1 \end{vmatrix} = (-f_y + g_y)\hat{i} - (-f_x + g_x)\hat{j} + (f_x g_y - f_y g_x)\hat{k}$$

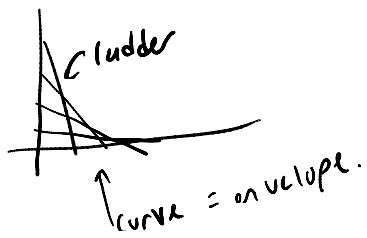
$$\langle -f_y + g_y, -f_x + g_x, f_x g_y - f_y g_x \rangle$$

Should pt. in direction of $\vec{T}$!

$$\left\langle 1, \frac{f_x - g_x}{g_y - f_y}, \frac{g_y f_x - f_y g_x}{g_y - f_y} \right\rangle$$

$= (g_y - f_y)\,\vec{T}$ direction.

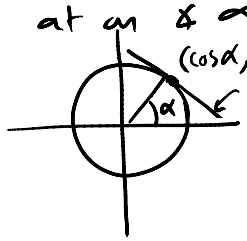
<u>Next Topic</u>: Envelopes



Line tangent to circle

Thursday, November 16, 2006
10:03 AM

Find equation of the line tangent to the circle $x^2+y^2=1$ at an α.



$(\cos\alpha, \sin\alpha)$ b/c $r=1$ in a unit circle.

$\frac{dy}{dx}$ = slope

$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Point slope

$$-\frac{\cos\alpha}{\sin\alpha} = \frac{y-\sin\alpha}{x-\cos\alpha}$$
$$-x\cos\alpha + \cos^2\alpha = y\sin\alpha - \sin^2\alpha$$
$$1 - x\cos\alpha = y\sin\alpha$$
$$1 = x\cos\alpha + y\sin\alpha$$

If you have a value α, you can find the tangent line.

Diff method

$\vec{r} = \langle \cos\alpha, \sin\alpha \rangle$

$\vec{T} = \langle -\sin\alpha, \cos\alpha \rangle$

$\langle x, y \rangle = \langle \cos\alpha, \sin\alpha \rangle + t\langle -\sin\alpha, \cos\alpha \rangle$ Parametric representation of the line.

$x = \cos\alpha - t\sin\alpha$ $\quad t = \frac{-x}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha}$

$y = \sin\alpha + t\cos\alpha$

$y = \sin\alpha + \left(\frac{-x}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha}\right)\cos\alpha$

$y = \sin\alpha - \frac{x\cos\alpha}{\sin\alpha} + \frac{\cos^2\alpha}{\sin\alpha}$

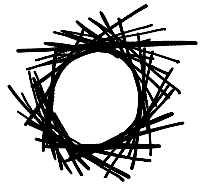
$y\sin\alpha = \sin^2\alpha - x\cos\alpha + \cos^2\alpha$

$y\sin\alpha = 1 - x\cos\alpha$

FOX MULDER

DANA SCULLY

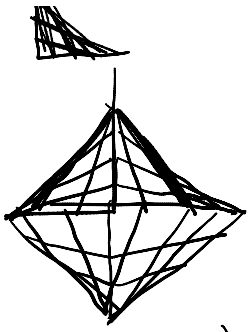
Here we consider a 1 parameter family of curves in the plane
(last example: α = parameter (only one parameter))
curves = straight lines tangent to the circle.



Envelope is a single curve in the plane that has the following properties: touches the circle, & is tangent to it.

Tangent to every curve in 1 parameter family.

Define family of curves by $F(x, y, \alpha) = 0$



Previous example: $F = x\cos\alpha + y\sin\alpha - 1$

Let $\vec{r} = \langle x(\alpha), y(\alpha)\rangle$ = envelope

Evelope satisfies

① $F(x(\alpha), y(\alpha), \alpha) = 0$

② Slope of tangent lines are the same

$F(x,y,\alpha) = 0 \leftarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$ b/c $\frac{dF}{\partial x} + \frac{\partial F}{\partial y} = 0$

$\vec{F} = \langle x(\alpha), y(\alpha)\rangle$

$\vec{T} = \langle \frac{dx}{d\alpha}, \frac{dy}{d\alpha}\rangle \leftarrow \frac{dy}{dx} = \frac{\left(\frac{dy}{d\alpha}\right)}{\left(\frac{dx}{d\alpha}\right)}$

$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{\left(\frac{dy}{d\alpha}\right)}{\left(\frac{dx}{d\alpha}\right)}$

$-F_x \frac{dx}{d\alpha} = F_y \frac{dy}{d\alpha} \rightarrow F_x \frac{dx}{d\alpha} + F_y \frac{dy}{d\alpha} = 0$

Aside: $\frac{d}{dx}\left[F(x(\alpha), y(\alpha), \alpha)\right] = 0$

$F_x \frac{dx}{d\alpha} + F_y \frac{dy}{d\alpha} + F_\alpha = 0$

For our envelope, the slopes of the tangent lines are the same & therefore their differences are 0.

$F(x,y,\alpha) = 0$ Envelope touches family of curves

$F_x(x,y,\alpha) = 0$

<u>ex</u> Find envelope for the family of lines $F = x\cos\alpha + y\sin\alpha - 1 = 0$

$F_\alpha = \frac{\partial F}{\partial \alpha} = -x\sin\alpha + y\cos\alpha = 0$

Show that this is envelope equation

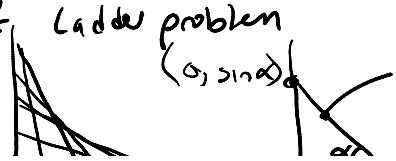
$x\cos\alpha + y\sin\alpha = 1 \rightarrow x^2\cos^2\alpha + 2xy\cos\alpha\sin\alpha + y^2\sin^2\alpha = 1$

$-x\sin\alpha + y\cos\alpha = 0 \rightarrow x^2\sin^2\alpha - 2xy\cos\alpha\sin\alpha + y^2\cos^2\alpha = 0$

add

$\boxed{x^2 + y^2 = 1}$ which is a unit circle.

<u>ex2</u> Ladder problem

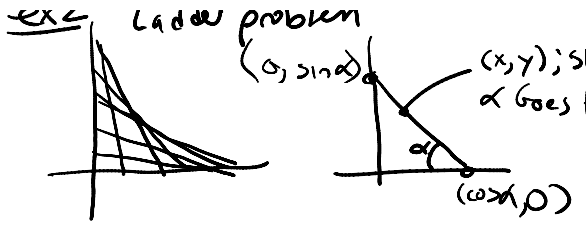


(x,y); slope is tangent of $\alpha = -\tan\alpha$ b/c $\frac{dy}{dx}$

α Goes from $\frac{\pi}{2} \rightarrow 0$

$\alpha \in [0, \frac{\pi}{2}]$

ex: Ladder problem



b/c $\frac{dy}{dx}$

$$\alpha \in \left[0, \frac{\pi}{2}\right]$$

$$\frac{y}{x-\cos\alpha} = -\tan\alpha$$

$$y = -\frac{\sin\alpha}{\cos\alpha}x + \frac{\sin\alpha}{\cos\alpha}\cos\alpha$$

$$y\cos\alpha + x\sin\alpha = \sin\alpha\cos\alpha$$

$$F = y\cos\alpha + x\sin\alpha - \sin\alpha\cos\alpha = 0$$

$$F_\alpha = -y\sin\alpha + x\cos\alpha - \cos^2\alpha + \sin^2\alpha = 0$$

$$y = \frac{x\cos\alpha}{\sin\alpha} - \frac{\cos^2\alpha}{\sin\alpha} + \sin\alpha$$

$$F = \frac{x\cos^2\alpha}{\sin\alpha} - \frac{\cos^3\alpha}{\sin\alpha} + \sin\alpha\cos\alpha + x\sin\alpha - \sin\alpha\cos\alpha = 0$$

$$F = x\left[\cos^2\alpha + \sin^2\alpha\right] = \cos^3\alpha \Rightarrow x = \cos^3\alpha$$

$$y = \frac{\cos^4\alpha}{\sin\alpha} - \frac{\cos^2\alpha}{\sin\alpha} + \sin\alpha = \frac{\cos^2\alpha\left(\cos^2\alpha - 1\right) + \sin^2\alpha}{\sin\alpha}$$

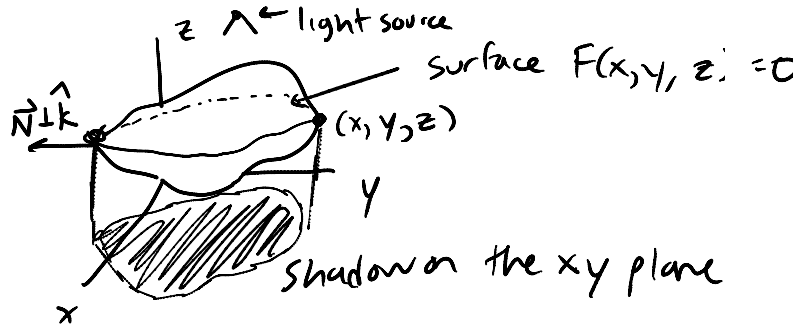
$$= \frac{\left(\cos^2\alpha - 1\right)^2}{\sin\alpha} = \frac{\sin^4\alpha}{\sin\alpha} = \sin^3\alpha$$

$$y = \sin^3\alpha$$

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

Shadow, Sonic Boom

Friday, November 17, 2006
10:02 AM



Equation for boundary of the shadow?

The normal vector at point (x,y,z) is always $\perp$ to the tangent at that pt, but at (x,y,z), it is also $\parallel$ to the xy plane. $\vec{N}$ is also $\perp$ to $\hat{k}$.

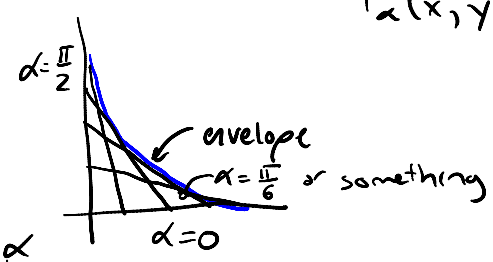
The equation for the boundary satisfies the following 2 eqns.

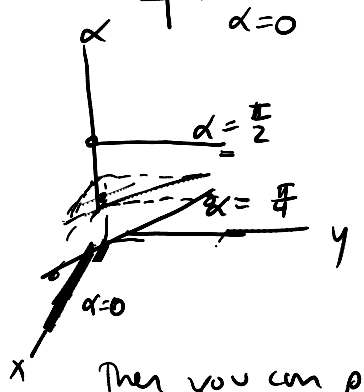
$F(x,y,z)=0$ Point has to be on the surface.

$$\hat{k}\cdot\nabla F = \hat{k}\cdot\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle = \frac{\partial F}{\partial z} = 0$$

$F(x,y,z)=0$
$F_z(x,y,z)=0$

Compare eqns for the envelope of a family of eqns. $F(x,y,\alpha)=0$
$F_\alpha(x,y,\alpha)=0$

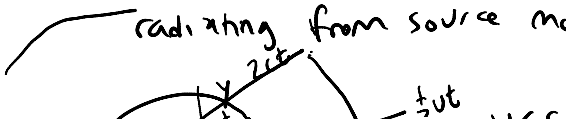


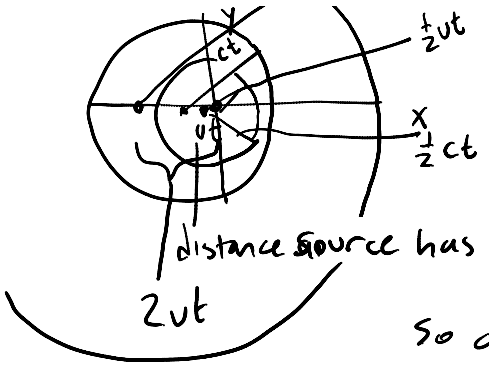


Then you can project the surface downward & get a shadow.

<u>Sonic Boom example</u>

Sound waves (with sound speed c) radiating from source moving w/ speed u



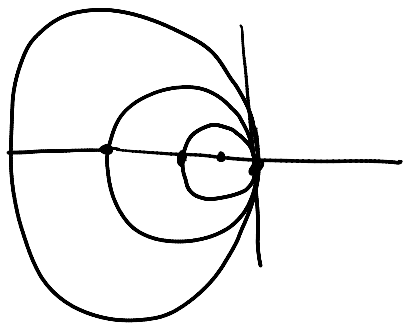


$v < c$

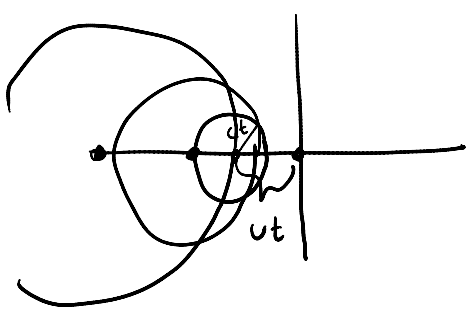
in frame of source, always at $x=y=z=0$

distance source has traveled for each vt, sound travels ct.

So as your time changes, your distance changes & your circles change. No envelope b/c you never catch up to the speed.



when $v=c$ no envelope



when $v > c$

$ct < vt$

this time, there's an envelope & it's a cone.

Families of surfaces = spheres.

$M = \text{Mach} = \frac{v}{c}$

Parameterize spheres of sound by time elapsed, t.

The origin of spheres is $-vt$ away from the origin b/c we're moving in the x direction.

Radius of the sphere $= ct$

$$(x+vt)^2 + y^2 + z^2 = c^2t^2$$

$$F\left((x+vt)^2 + y^2 + z^2 - c^2t^2 = 0\right)$$

$$F_t\left(2(x+vt)\cdot v - 2c^2t = 0\right)$$

$$2(x+vt)\cdot v - 2c^2t = 0$$

$$2vx + 2v^2t = 2c^2t$$

$$2vx = 2c^2t - 2v^2t$$

solve for t & plug into $F(x,y,z,t)=0$ to get envelope equation

$$2ux = t(2c^2 - 2u)$$

$$\boxed{\frac{ux}{c^2-u^2} = t}$$

$$F\left(\left(x + u\left(\frac{ux}{c^2-u^2}\right)\right)^2 + y^2 + z^2 - c^2\left(\frac{ux}{c^2-u^2}\right)^2\right) = 0$$

$$F = x^2 + \frac{2x^2u^2}{c^2-u^2} + \frac{u^2x^2}{(c^2-u^2)^2} - y^2 + z^2 - \frac{c^2u^2x^2}{(c^2-u^2)^2}) = 0$$

$$F = x^2\left[\frac{(c^2-u^2+u^2)^2}{(c^2-u^2)^2} - \frac{c^2u^2}{(c^2-u^2)^2}\right] + y^2 + z^2 = 0$$

$$F = x^2\left[\frac{c^4-c^2u^2}{(c^2-u^2)^2}\right] + y^2 + z^2 = 0$$

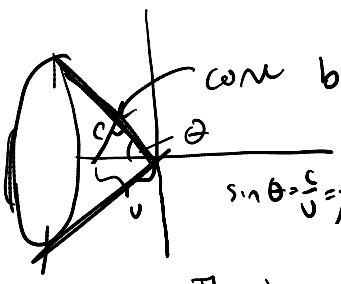
$$F = \frac{x^2c^2}{c^2-u^2} + y^2 + z^2 = 0$$

$$y^2 + z^2 = \frac{x^2c^2}{u^2-c^2}$$

$$= \frac{x^2c^2}{M^2c^2-c^2} = \frac{x^2}{M^2-1}$$

$$y^2 + z^2 = \frac{x^2}{M^2-1}$$

There is no surface if the mach # is <1. $M<1$, $\emptyset$. $M=1$, ??
$M>1$, cone.



cone b/c for each point, it's a circle.

$\sin\theta = \frac{c}{u} = \frac{1}{M}$

The pilot doesn't hear anything, so neither does the guy next to him (in another plane)

The bigger the Mach #, the smaller the sin of the ∡ is. The cone widens as you slow down.

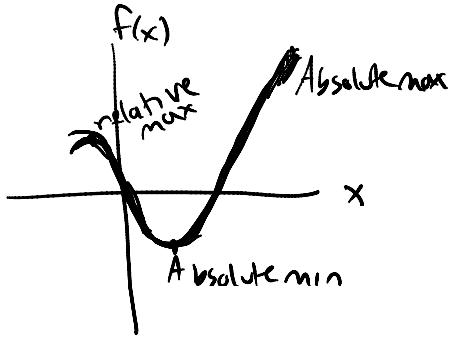
Max/Min Problems

1D Find the max & min of the following function if $f(x) = x^3 - 12x$
$-3 \le x \le 5$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$0 = 3(x^2 - 4)$$
$$4 = x^2$$
$$\pm 2 = x$$

f': $+$ | $-$ | $+$ at -3, -2, 2, 5

We should check $-3, -2, 2, 5$ for C.P. (i.e., max/min)



$f(-3) = -27 + 36 = 9$
$f(-2) = -8 + 24 = 16$
$f(2) = 8 - 24 = -16$ ← absolute min
$f(5) = 125 - 60 = 65$ ← absolute max

For a continuous & differentiable function $y = f(x)$
1) At a local max/min (or relative max/min) $f'(x_0) = 0$ $x = x_0$, $y = f(x_0)$
2) A function on an interval I obtains its absolute max/min either i) at an interior pt to I where $f' = 0$
ii) at a pt. on the boundary of the interval I.

2D Suppose we have a surface $z = f(x,y)$ (that is continuous & differentiable)

Both partial derivatives $= 0$. Instead of a tangent line, we have a tangent plane w/ respect to the z axis, horizontal.

At a local max/min, $f_x = f_y = 0$ This is equivalent to stating that the tangent plane is horizontal.

Proof

$$dz = f_x\,dx + f_y\,dy$$
$$\Delta z \approx f_x\,\Delta x + f_y\,\Delta y$$

Assume we have a local max at some pt (x_0, y_0).
Proof by contradiction. Assume f_x at $(x_0, y_0) \neq 0$. Increase height z by Δz such that $f_x\,\Delta x$ is pos.

$$\Delta z = f_x \Delta x > 0 \Rightarrow (x_0, y_0) \not\in \text{local max}$$

contradiction to original claim.

original claim.

If you can show the normal vector is vertical, $\perp$ to tangent plane, you know you've found a horizontal tangent plane.

$$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$$

Recall a normal vector to surface at that pt

surface $\vec{z} = (x,y)$ $\quad \vec{n} =$

$F(x,y,z) = z - f(x,y) = 0 \leftarrow \nabla \vec{F} \perp$ to $F=0$

$\nabla F = \langle -f_x, -f_y, 1 \rangle$

And so it's vertical $\rightarrow$ the normal vector is $\perp$ to tangent plane & vertical.

<u>Ex</u>

Find places where tangent plane is horizontal to surface

$$f(x,y) = x^3 + 3xy^2 - 3x$$

$$f_x = 3x^2 + 3y^2 - 3 = 0$$

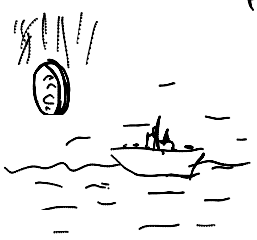
$$f_y = 6xy = 0 \Leftarrow x=0,\ y=0$$

$$3y^2 - 3 = 0 \qquad 3x^2 - 3 = 0$$

$$y = \pm 1 \qquad x = \pm 1$$

Horizontal tangent plane at 4 places

$(x,y) = (0,1), (0,-1), (1,0), (-1,0)$



A function $z = f(x,y)$ over a region R contains its absolute at 1) A pt. interior to R where $f_x = f_y = 0$

or 2) On the boundary of R $\quad$ boundary of $R = (\partial R)$

<u>ex</u>. Find the absolute max/min of $f = x^2 + 3y^2$, $x^2+y^2 \le 1$

Check interior $f_x = 2x = 0$ $\quad (x,y) = (0,0)$

$f_y = 6y = 0$

The smallest this function could be is zero b/c it's the sum of squares. So $(x,y) = (0,0)$ is a min.

Max is on boundary where $x^2+y^2 = 1$

let $x = \cos\theta$ $\quad y = \sin\theta$

$f = \cos^2\theta + 3\sin^2\theta$ $\quad$ find θ that gives largest f.

$= 1 + \underbrace{2\sin^2\theta}_{=1 \text{ is largest value.}}$

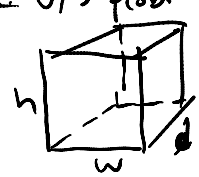
$\sin^2\theta = 1$

$\pm\frac{\pi}{2} = \theta$ $\quad$ Absolute max is at

$(x,y) = (0, \pm 1)$

$(x,y) = (0, \pm 1)$ at

Ex. UPS prob.



Largest volume box subject to the constraint that $h+w+d \leq 90\text{ in} = \ell$

Answer is 30x30x30 b/c symmetry = largest volume

$V = whd$

subject to $\ell = h+w+d$

solve for d

$\ell - h - w = d$

$V = wh(\ell - h - w)$ so now it's just a function of w & h

$\frac{\partial V}{\partial h} = w(\ell - h - w) + wh(\ell - w) = w(\ell - w - 2h)$

$\frac{\partial V}{\partial w} = h(\ell - h - w) + wh(\ell - h) = h(\ell - 2w - h)$

At max volume, $w(\ell - w - 2h) = 0$
$h(\ell - 2w - h) = 0$

$\ell - w - 2h = 0$ 2 eqns, 2 unknowns
$\ell - 2w - h = 0$

2nd Derivative Test in Taylor Series

Tuesday, November 21, 2006
9:57 AM

2nd Derivative Test in terms of Taylor Series

function of variable

Use Taylor series

$y = f(x) \qquad f'(x_0) = 0$

x_0 = extremum (possible max, min, or neither)

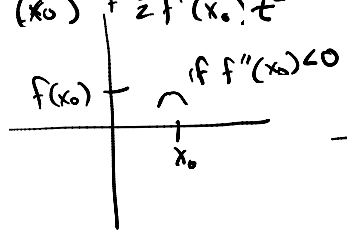
expand $f(x)$ about $x = x_0$

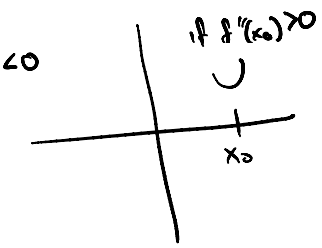
$$f(x) = f(x_0 + t) = f(x_0) + \cancel{f'(x_0)}^{\,0}\, t + \frac{1}{2} f''(x_0)\, t^2 + E$$

expanding about $t = 0$

$$|E| < C t^3$$

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)\, t^2$$





Mulder: Sorry, no one down here but the FBI's most UN wanted

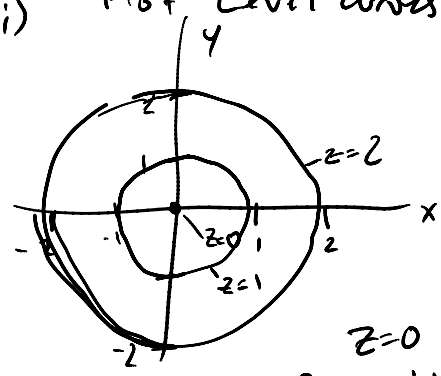
Scully: Agent Mulder? My name's Dana Scully. I've been assigned to work with you.

Mulder: Oh, isn't it nice to suddenly be so highly regarded. So who did you kick off to get this assignment, Scully?

Scully: Actually, I'm looking forward to working with you. I've heard a lot about you.

Mulder: Oh, really? I was under the impression you were sent to spy on me.

i) Plot Level curves



$z = x^2 + y^2$

$z = 0$ is global minimum

<u>Second Derivative Test</u>

ex. (i) $z = x^2 + y^2 = f(x,y)$

(ii) $z = -x^2 - y^2 = f(x,y)$

(iii) $z = xy, = f(x,y)$

i) $f_x = 2x = 0 \quad f_y = 2y = 0$

$x = y = 0$ is extremum

ii) $f_x = -2x = 0 \quad f_y = -2y = 0$

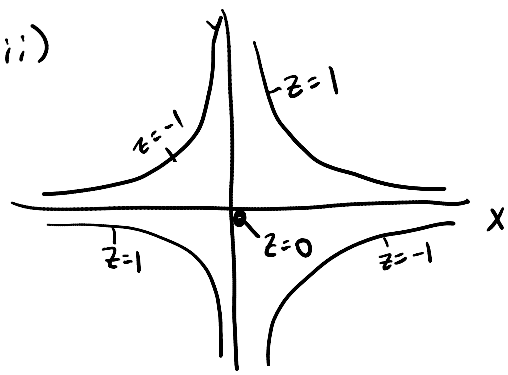
extremum at origin

iii) $f_x = y = 0, \; f_y = x = 0$

extremum at origin.

These extremums will be max, min

iii)



$z = xy$

Saddle

Taylor expand $z = f(x,y)$ about an extremum, $(x,y) = (a,b)$

$$f_x(a,b) = f_y(a,b) = 0$$

$f(x,y) = f(a+h, b+k)$ $\qquad |h|, |k| \ll 1$

Trick: Taylor expand about $t=0$

$$f(x,y) = F(a+ht, b+kt)$$

$x = a+ht$

$y = b+kt$

$$F(t) = F(0) + F'(0)t + \tfrac{1}{2}F''(0)t^2 + \mathcal{E}$$

$$F(0) = f(a,b)$$

$$F'(t)\Big|_{t=0} = \frac{d}{dt}\left[f(a+ht, b+kt)\right] = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\Bigg|_{t=0}$$

$$= f_x(a,b)\,h + f_y(a,b)\,k$$

$$F''(t)\Big|_{t=0} = \frac{d}{dt}\left[\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k\right]$$

$$= \Big(f_{xx}\underbrace{\frac{dx}{dt}}_{h} + f_{xy}\underbrace{\frac{dy}{dt}}_{k}\Big)h + \Big(f_{yx}\underbrace{\frac{dx}{dt}}_{h} + f_{yy}\underbrace{\frac{dy}{dt}}_{k}\Big)k$$

$$f(x,y) = f(a+h, b+k) \leftarrow t=1$$

$$f(a,b) = f(a,b) + \left[f_x(a,b)h + f_y(a,b)k\right]$$
$$+ \tfrac{1}{2}\left[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}k^2\right]$$
$$+ E, \quad |E| < Ch^3 + C_2h^2k + C_3hk^2 + C_4k^3$$

Extremum at $(x,y) = (a,b)$

$$f_x(a,b) = f_y(a,b) = 0$$

$$f(x,y) \approx f(a,b) + \tfrac{1}{2}\left[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}k^2\right]$$

No matter what choice of h + k, the function does what it's supposed

No matter what choice of h + k, the function does what it's supposed to (increase, decrease, etc)

$$= f(a,b) + \frac{1}{2}\left[f_{xx}(a,b)\left(h + \frac{f_{xx}(a,b)}{f_{xx}(a,b)}k\right)^2 - f_{xx}(a,b)\frac{f_{xy}^2(a,b)}{f_{xy}^2(a,b)}k^2 + f_{yy}(a,b)k^2\right]$$

$$= f(a,b) + \frac{1}{2}\left[f_{xx}(a,b)\left(h + \frac{f_{xx}(a,b)}{f_{xx}(a,b)}k\right)^2 + \frac{k^2}{f_{xx}(a,b)}\left(f_{yy}(a,b)f_{xx}(a,b) - f_{xy}^2(a,b)\right)\right]$$

$$= f(a,b) + \frac{1}{2}\frac{1}{f_{xx}(a,b)}\left[\underbrace{\left(f_{xx}(a,b)h + f_{xy}(a,b)k\right)^2}_{\geq 0} + \underset{\geq 0}{k^2}\underbrace{\left(f_{yy}(a,b)f_{xx}(a,b) - f_{xy}(a,b)^2\right)}_{?}\right]$$

If $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$ & $f_{xx} < 0$
Then it's a maximum b/c you're always subtracting something.

If $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$ & $f_{xx} > 0$
Then it's a minimum b/c you're always adding something

If $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) < 0$ then you can choose h & k so that it's either negative or positive. Saddle situation.

LaGrange Multiples, Constrained Optimization

Monday, November 27, 2006
10:04 AM

Look for places the tangent plane is horizontal. For that to occur, ∇F is 0

$z = f(x,y)$ At an extremum, (critical pt.) $(x,y) \in (a,b)$

$$\frac{\partial f}{\partial x}_{(a,b)} = \frac{\partial f}{\partial y}_{(a,b)} = 0$$

Taylor Series of $f(x,y)$ about (a,b)

$$f(a+h, b+k) = f(a,b) + \left[\cancelto{0}{f_x(a,b)}\, h + \cancelto{0}{f_y(a,b)}\, k\right] + \frac{1}{2}\left[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2\right] + E$$

0 b/c (a,b) is critical pt

E is cubic in h & k

$$f(a+h, b+k) - f(a,b) \approx \frac{1}{2}\,\frac{1}{f_{xx}}\left[\underbrace{(f_{xx}h + f_{yy}k)^2}_{\geq 0} + \underbrace{k^2}_{\geq 0}(f_{xx}f_{yy} - f_{xy}^2)\right]$$

$f_{xx}f_{yy} - f_{xy}^2 > 0$ Then the whole thing is ≥ 0.

THEN

If $f_{xx} > 0$ (a,b) is local min

If $f_{xx} < 0$ (a,b) is local max

B/c everywhere you go, you are increasing/decreasing from that pt.

$f_{xx}f_{yy} - f_{xy}^2 < 0$ Then depending on k^2's size, it could be pos. or neg. This corresponds to the saddle case.

And then it depends on f_{xx} again.

predict We got this whole thing from Taylor Series

		f_x	f_y	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	
min	$f = x^2 + y^2$	$2x$	$2y$	2	2	0	4	$f_{xx} > 0 \Rightarrow$ min
max	$f = -(x^2+y^2)$	$-2x$	$-2y$	-2	-2	0	4	$f_{xx} < 0 \Rightarrow$ max
saddle	$f = x^2 - y^2$	$2x$	$-2y$	2	-2	0	-4	saddle
saddle	$f = xy$	y	x	0	0	1	-1	saddle

The only extremum is at the origin

Find & classify all critical points of $f(x,y) = 3xy - x^2y - xy^2$

$$\frac{\partial f}{\partial y} = 3y - 2xy - y^2$$

$$\frac{\partial f}{\partial x} = 3x - x^2 - 2xy$$

∂y - ... ...,

$3y - 2xy - y^2 = 0$ $\qquad$ $3x - x^2 - 2xy = 0$

At $y=0$, it's 0

$y(3-2x-y)=0$ $\qquad$ $x(3-x-2y)=0$

$\Rightarrow y=0$, or $\qquad$ $\Rightarrow x=0$, or

$3-2x-y=0$ $\qquad$ $3-x-2y=0$

so 4 critical pts.

$y=0$, $x=0$ or $x=3$ $\qquad$ $3-2x-y=0$, $x=0$ or $3-x-2y=0$

$(0,0)$ or $(3,0)$ $\qquad$ $y=3$ $\qquad$ $(4,1)$

$(0,3)$ or $(1,1)$

Remember this

f	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy}-f_{xy}^2$	
$(0,0)$	$-2y=0$	$-2x=0$	$3-2x-2y=3$	-9	saddle
$(3,0)$	$-2y=0$	$-2x=-6$	$3-2x-2y=-3$	-9	saddle
$(0,3)$	$-2y=-6$	$-2x=0$	$3-2x-2y=-3$	-9	saddle
$(1,1)$	$-2y=-2$	$-2x=-2$	$3-2x-2y=-1$	3	$f_{xx}<0$ so max (local)

No local min. in this function

Is $(1,1)$ a global max?

$3xy - x^2y - xy^2 = xy(3-x-y)$ $\qquad$ You can easily make x & y neg & huge, & so it's just a local max (you can make the function much larger by chosing x & y as really big & neg.)

<u>This</u> is the 2nd derivative test, right here.

Lagrange Multipliers & constrained optimization

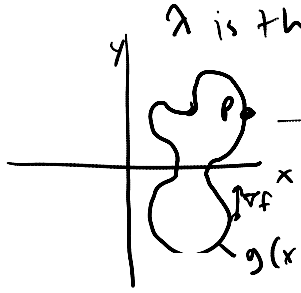
State theorem in 2 dimensions, 1 constraint

Let $f(x,y)$ & $g(x,y)$ be continuously differentiable functions

If the max or min value of $f(x,y)$ subject to the constraint $g(x,y)=0$ occurs at a point P where the gradient of g at $P \neq 0$ $(\nabla g(P) \neq 0)$

Then $\nabla f(P) = \lambda \nabla g$, where λ is some #.

λ is the Lagrange Multiplier.



$\nabla f(P) = \lambda \nabla g$

$\nabla g \perp$ to curve (b/c level curve) $\quad g=0$

directional derivative is dot product of tangent vector w/ this vector, so you can increase ∇f. when directional derivative is 0, you don't know whether to move right or left, b/c ∇f & $\vec{v}$ are $\perp$.

$g(x,y)=0$ defines curve in 2 dimensions.

$\nabla f \cdot \hat{v} = 0$

$g(x,y) = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \lambda \left.\frac{\partial g}{\partial x}\right|_{(x,y)}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)} = \lambda \left.\frac{\partial g}{\partial y}\right|_{(x,y)}$$

LaGrange Multipliers

Tuesday, November 28, 2006
9:57 AM

Lagrange Multipliers

2D function $f(x,y)$ that I want to extremize but we have the constraint $g(x,y)=0$

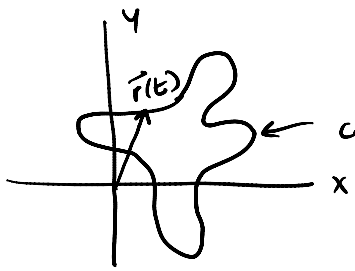
Soln. $g(x,y)=0$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Solve for λ, x, y And then values of x & y are candidates for max & min.

$$\nabla f = \lambda \nabla g$$



constraint $g(x,y)=0$

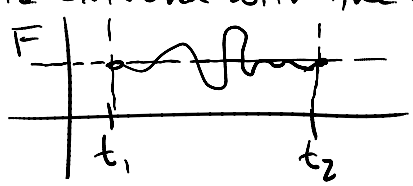
curve presented parametrically

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$t_1 \le t \le t_2$$

As you go around the curve you sweep it out w/ the vector.

Then find extrema with the function $F(t) = f(x(t), y(t))$



Calculate derivative of F w/ respect to t & find where it's zero.

$$\frac{dF}{dt} = 0 = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \vec{v}$$

where $\vec{v} = \frac{d\vec{r}}{dt}$

$\vec{v}$ is a tangent vector to the curve.

∇f is $\perp$ to tangent vector at extrema.

$$\nabla f \perp \hat{T}, \quad \nabla f \perp \vec{v}$$

Our curve is a level curve of the function $f(x,y)$

Generalizations

Function $f(x_1, \ldots x_n)$ of n variables subject to constraint $g(x_1 \ldots x_n) = 0$

Extrema satisfy
$$g(x_1, \ldots x_n) = 0$$
$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1}$$
$$\frac{\partial f}{\partial x_2} = \lambda \frac{\partial g}{\partial x_2}$$
$$\frac{\partial f}{\partial x_n} = \lambda \frac{\partial g}{\partial x_n}$$

You can have a problem w/ multiple constraints, though.
Function of n variables $(x_1, \ldots x_n)$ subject to k constraints
$$g_1(x_1 \ldots x_n) = 0$$
$$g_2(x_1 \ldots x_n) = 0$$
$$\vdots$$
$$g_k(x_1 \ldots x_n) = 0$$
Defines a surface w/ dimension $n-k$

Extrema satisfy
$$g_1(x_1 \ldots x_n) = 0$$
$$g_2(x_1 \ldots x_n) = 0$$
$$g_n(x_1 \ldots x_n) = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \ldots + \lambda_k \nabla g_k$$
You end up introducing k Lagrange multipliers.

<u>Ex</u> Find the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ that is furthest from $(1, 2, 3)$

You're trying to extremize distance.
$$F(x, y, z) = D^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$$
constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

4 eqns, 4 unknowns

$g = 0$ $\qquad x^2 + y^2 + z^2 - 1 = 0$

$\nabla f = \lambda \nabla g$ $\qquad 2(x-1) = \lambda 2x \Rightarrow x(1-\lambda) = 1 \quad x = \frac{1}{1-\lambda}$

$2(y-2) = \lambda 2y \quad y(1-\lambda) = 2 \quad y = \frac{2}{1-\lambda}$

$2(z-3) = \lambda 2z \quad z(1-\lambda) = 3 \quad z = \frac{3}{1-\lambda}$

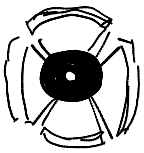
$$\underbrace{\quad}_{\nabla f} \quad \underbrace{\quad}_{\lambda \nabla g}$$

$$x^2+y^2+z^2 = \left(\frac{1}{1-\lambda}\right)^2(1+4+9) = 1$$

$$(1-\lambda)^2 = 14$$

$$1-\lambda = \pm\sqrt{14}$$

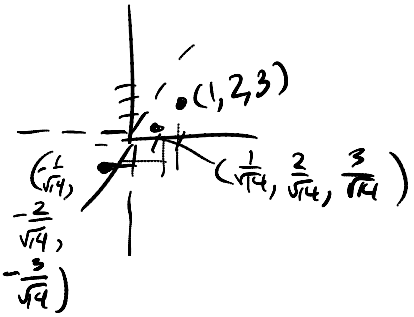
$$1 \pm \sqrt{14} = \lambda$$



$$x = \pm \frac{1}{\sqrt{14}}$$

$$y = \pm \frac{2}{\sqrt{14}}$$

$$z = \pm \frac{3}{\sqrt{14}}$$



Max & Min.

Alternative statement of Lagrange Multipliers

Extremize $f(x,y,z)$ subject to constraint $g(x,y,z)=0$

by finding extrema of $F(x,y,z,\Lambda) = f(x,y,z) + \Lambda g(x,y,z)$

4 partial derivatives = to 0.

$$\left.\begin{aligned} 0 &= \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \Lambda \frac{\partial g}{\partial x} \\ 0 &= \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \Lambda \frac{\partial g}{\partial y} \\ 0 &= \frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \Lambda \frac{\partial g}{\partial z} \end{aligned}\right\} \Lambda = \lambda$$

$$0 = \frac{\partial F}{\partial \Lambda} = g(x,y,z)$$

constraint

$\nabla f = \lambda \nabla g$